2-nondegenerate CR structures of hypersurface type: bigraded Tanaka prolongation for canonical absolute parallelism and maximally symmetric models.

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CR-structures of hypersurface type

M is a real hypersurface in $\mathbb{C}^{n+1}$,

$D(q) = T_qM \cap \mathbf{i}T_qM$ is the maximal complex subspace of T_qM;

$J : D \to D$ be the restriction of multiplicaton by i to D, $J^2 = -\mathrm{Id}$;

The pair (D, J) defines the *C*R structure on the real hypersurface M;

The $\mathbf{i}$-eigenspace of $H \subset \mathbb{C}D$ of J is called the *holomorphic subbundle* of $\mathbb{C}TM$;

Integrability condition: $[H, H] \subset H$.

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Levi kernel of CR structure

Levi form is an Hermitian form on H:

$$\mathfrak{L}(X,Y) = \mathbf{i}[X,\overline{Y}] \mod \mathbb{C}D.$$

Levi kernel $K = \ker\mathfrak{L}$.

Levi nondegenerate case, i.e. when $K = 0$, is very well understood case (Cartan, Tanaka, Chern-Moser).

Uniformly Levi degenerate case is when K is non-trivial at every point.

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Levi 2-nondegenerate structures

Fix $x \in M$. For any $v \in K$ define

$$\begin{aligned} \mathrm{ad}_v : \overline{H}_x/\overline{K}_x &\to H_x/K_x, \\ y &\mapsto [V,Y]|_x \mod K_x \oplus \overline{H}_x, \end{aligned}$$

where V and Y are extensions of v and y to local section of K and $\overline{H}/\overline{K}$, respectively.

Definition

A CR structure is called 2-nondegenerate if $\mathrm{ad}_v \neq 0$ for all $v \neq 0$.

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Previously known results

The smallest dimension of M when 2-nondegenericity may occur is $\dim_{\mathbb{R}} M = 5 \Rightarrow \dim_{\mathbb{C}} K = 1$:

$$\dim_{\mathbb{C}} H > \dim_{\mathbb{C}} K \geq 1 \Rightarrow \dim_{\mathbb{C}} H \geq 2 \Rightarrow \dim_{\mathbb{R}} M \geq 5.$$

Theorem (Isaev-Zaitsev (2013), Pocchiola (2013), Medori-Spiro (2014))

For $\dim M = 5$ to any 2-nondegenerate CR structure of hypersurface type one can assign the canonical absolute paralllelism on 10-dimensional bundle over M and there exists the unique, up to local diffeomorphism, maximally symmetric model with the algebra of infinitesimal symmetries $\mathfrak{so}(2,3)$.

C. Porter (2016) – similar results in $\dim_{\mathbb{C}} M = 7$, $\dim_{\mathbb{C}} K = 1$ under some additional algebraic assumptions.

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The scope of our results

C. Porter, I.Z. (2017)-bigraded analog of Tanaka prolongation for construction of absolute paralellism for 2-nondegenerate CR structures of hypersurface type of arbitrary odd dimension under additional algebraic assumptions (which even in dimension 7 are weaker than of Porter 2016).

Tanaka like bigraded prolongation for 2-nondegenerate CR structures of hypersurface type

We work with complexified object: natural filtration on $\mathbb{C}TM$:

$$K \oplus \bar{K} \subset \mathbb{C}D \subset \mathbb{C}TM$$

Associated grading:

$$\underbrace{K \oplus \bar{K}}_{\text{of weight } 0} \oplus \underbrace{\mathbb{C}D/(K \oplus \bar{K})}_{\mathfrak{g}_{-1}} \oplus \underbrace{\mathbb{C}TM/\mathbb{C}D}_{\mathfrak{g}_{-2}},$$

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Natural bigrading

- $\mathfrak{g}_{-1,-1} := \bar{H}/\bar{K}$, $\quad \mathfrak{g}_{-1,1} := H/K$, $\quad \mathfrak{g}_{-2,0} := g_{-2}$;
- $\forall v \in K$: $\mathrm{ad}_v : \underbrace{\overline{H}/\overline{K}}_{\mathfrak{g}_{-1,-1}} \to \underbrace{H/K}_{\mathfrak{g}_{-1,1}}$

  Extend ad_v trivially to $H/K (= g_{-1,1})$ and to $\mathfrak{g}_{-2} (= \mathfrak{g}_{-2,0})$: $\mathrm{ad}_v|_{\mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{-2,0}} = 0$. So, ad_v is identified with an element of $\mathrm{Der}(\mathfrak{g}_-) \cong \mathrm{csp}(\mathfrak{g}_-)$.
- $\mathfrak{g}_{0,2} :=$ the image in $\mathrm{Der}(\mathfrak{g}_-)$ of $\{\mathrm{ad}_v : v \in K\}$ under this identification;

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Definition

The symbol of a 2-nondegenerate CR structure of hypersurface type at a point is a bigraded vector subspace

$$\mathfrak{g}^0 = \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$$

of the Lie algebra $\mathfrak{g}_- \oplus \mathrm{Der}(\mathfrak{g}_-)$ together with the antilinear involution $^-$ induced by the complex conjugation on $\mathbb{C}D$: $\bar{A}v := A\bar{v}, A \in \mathrm{Der}(\mathfrak{g}_-)$.

Important point In general the symbol $\mathfrak{g}^0$ is not a Lie subalgebra of $\mathfrak{g}_- \oplus \mathrm{Der}(\mathfrak{g}_-)$: the operation of Lie brackets is compatible with the bigrading for all pairs of bigraded component except $(\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2})$, i.e. in general $[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}] \not\subseteq \mathfrak{g}_{0,0}$, because $\left[[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}], \mathfrak{g}_{0,\pm 2}\right] \not\subseteq \mathfrak{g}_{0,\pm 2}$

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$$[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}] \subset \mathfrak{g}_{0,0}$$

or equivalent $\mathfrak{g}^0$ is a Lie subalgebra of $\mathfrak{g}_- \oplus \mathrm{Der}(\mathfrak{g}_-)$.

Universal bigraded prolongation of CR symbol

Among all bigraded Lie algebras of the form

$$\mathfrak{g}^0 \oplus \underbrace{\mathfrak{g}_{1,-1} \oplus \mathfrak{g}_{1,1}}_{\text{no } g_{1,\pm 3} \text{ in } \mathfrak{g}_1} \oplus \bigoplus_{i \geq 2, j \in \mathbb{Z}} \mathfrak{g}_{i,j}$$

take the maximal non-degenerate one (non-degenerate means that for every nonzero x with non-negative first weight $\mathrm{ad}_x|_{\mathfrak{g}_-} \neq 0$).
This algebra is called the universal bigraded algebraic prolongation of $\mathfrak{g}^0$ and it is denoted by $\mathfrak{U}(\mathfrak{g}^0)$.

$$\mathfrak{g}_{1,-1} = \{f \in \widetilde{\mathfrak{g}}_{1,-1} \mid [f, \mathfrak{g}_{0,-2}] = 0, \big[[f, g_{0,2}], g_{0,2}\big] = 0\},$$
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where $\widetilde{\mathfrak{g}}_{1,1}, \widetilde{\mathfrak{g}}_{1,-1}$ come from the standard Tanaka prolongation of $\mathfrak{g}^0$.

It is endowed with the natural involution induced by the complex conjugation in $\mathbb{C}D$. Let $\Re\mathfrak{U}(\mathfrak{g}^0)$ be the real part of $\mathfrak{U}(\mathfrak{g}^0)$ with respect to this involution.

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The main theorem on existence of absolute parallelism

Theorem (a bigraded analog of the standard Tanaka theorem)

Assume that $\dim \mathfrak{U}(\mathfrak{g}^0) < \infty$.

1. *To any* 2*-nondegenerate, hypersurface type CR structure with regular symbol* $\mathfrak{g}^0$ *one can assign the canonical structure of absolute parallelism on a bundle over* M *of (real) dimension equal to* $\dim_{\mathbb{C}} \mathfrak{U}(\mathfrak{g}^0)$;
2. *Up to a local diffeomorphism , there exists the unique maximally symmetric CR structure among all* 2*-nondegenerate CR structures with constant symbol* $\mathfrak{g}^0$ *and its algebra of infinitesimal symmetries is isomorphic to the real part* $\Re\mathfrak{U}(\mathfrak{g}^0)$ *of* $\mathfrak{U}(\mathfrak{g}^0)$;

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Classification of regular symbols with one dimensional Levi kernel

The space of symbols of 2-nondegenerate CR structures, up to an isomorphism $\cong$ the space of pairs

(a real line ℓ of nondegenerate Hermitian forms on $\mathfrak{g}_{-1,1}$, a complex line of self-adjoint anti-linear operators A on $\mathfrak{g}_{-1,1}$),

up to the natural action of $GL(\mathfrak{g}_{-1,1})$, where

$$A(y) = \mathrm{ad}_v(\bar{y}), \quad v \in \mathfrak{g}_{0,2},\ y \in \mathfrak{g}_{-1,1}$$

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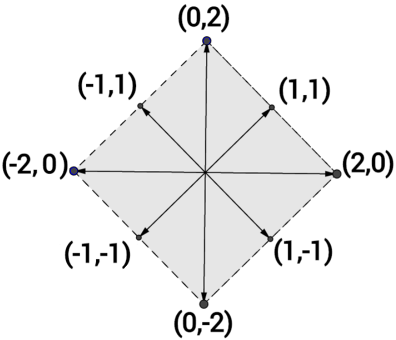
For $\dim_{\mathbb{R}} M = 5$, then $\dim_{\mathbb{C}} \mathfrak{g}_{-1,1} = 1$, so there is only one symbol and it is regular. $\mathfrak{U}(\mathfrak{g}^0) \cong \mathfrak{so}(5) \cong B_2, \quad \Re\mathfrak{U}(\mathfrak{g}^0) \cong \mathfrak{so}(3,2)$



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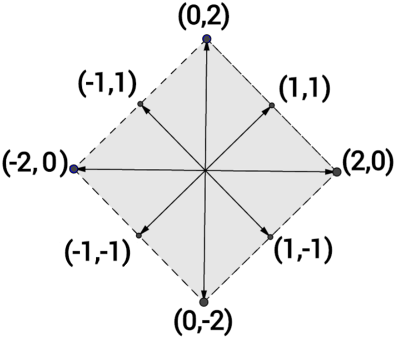

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Proposition

A symbol of 2-nondegenerate CR structure given by the pair $(\mathbb{R}\ell, \mathbb{C}A)$ is regular if and only if

$$A^3 = \alpha A, \quad \alpha \in \mathbb{R}.$$

We subdivide the set of regular 2-nondegenerate symbols with 1-dimensional Levi kernel (which is a discrete set) into

- *nilpotent regular* , if $\alpha = 0$ or , equivalently, $A^3 = 0$;
- *non-nilpotent regular* otherwise, i.e. when $\alpha \neq 0$. The latter type is subdivided into two subtypes:
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We calculated the universal bigraded prolongation for all such symbols.

- The third prolongation (w.r.t. the first weight) is always zero.
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Bigraded prolongations of strongly non-nilpotent symbols

α depends on the choice of a generator A of the corresponding line, but its sign is independent of this choice.

Assume that g^0 is strongly non-nilpotent regular symbol, described by the pair (ℓ, A). In this case $A^2 = \alpha I, \ \alpha \neq 0$. Let $\dim_{\mathbb{R}} M = 2n + 3$. Then

$$\mathfrak{U}(\mathfrak{g}^0) \cong \mathfrak{so}(n+4, \mathbb{C})$$

As $\Re\mathfrak{U}(\mathfrak{g}^0)$ one can gets any real form of $\mathfrak{so}(n+4, \mathbb{C})$, except $\mathfrak{so}(n+4)$ and $\mathfrak{so}(n+3, 1)$:

- If $\alpha > 0$: ℓ may have an arbitrary signature (p, q) with $p + q = n$. Then
$$\Re\mathfrak{U}(\mathfrak{g}^0) \cong \mathfrak{so}(p+2, q+2);$$
- The case $\alpha < 0$ may happen if and only if $\dim M \equiv 3 \mod 4$ and signature with $p = q = \frac{\dim M - 3}{4}$. Then
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Bigraded prolongation of nilpotent symbols with $A^2 = 0$

Assume that $\mathfrak{g}^0$ is weakly nilpotent, i.e. $A^3 = 0$, $A \neq 0$.

The classification is by the Jordan normal form of A where all blocks are nilpotent and of the size not greater than 3 and at least one block is of size 2.

The maximally symmetric case (for the fixed $\dim M \geq 7$) is when there is only one nontrivial Jordan block and it is of size 2. In this case

$$\dim \mathfrak{U}(\mathfrak{g}^0) = \left(\frac{\dim M - 1}{2}\right)^2 + 7$$

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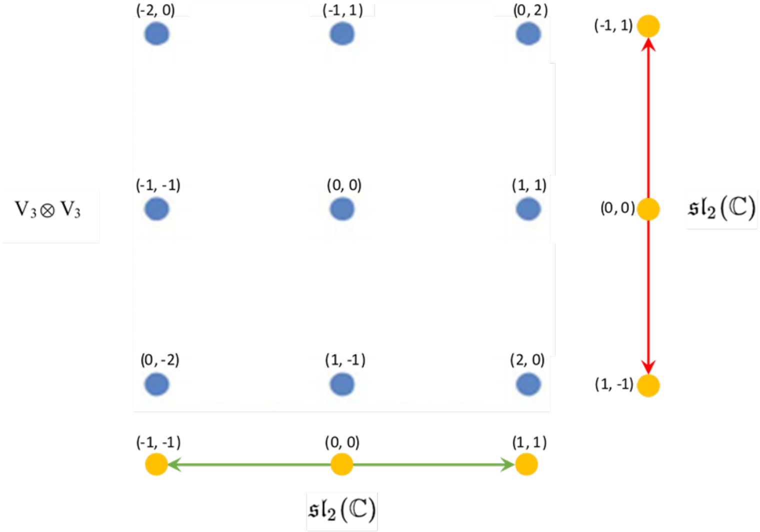
Comparing dimensions of prolongations for strongly non-nilpotent and maximally symmetric nilpotent cases

$\dim M$	$\dim \mathfrak{U}(\mathfrak{g}^0)$ for strongly non-nilpotent symbols	maximal $\dim \mathfrak{U}(\mathfrak{g}^0)$ for nilpotent symbols
7	15	16
9	21	23
11	28	32

For example, in the case $\dim M = 7$ and of $A^2 = 0$

$$\mathfrak{U}(g^0) = \Big(\mathbb{C} \oplus (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}))\Big) \ltimes (V_3 \otimes V_3)$$



Maximally homogeneous model for nilpotent symbols: hypersurface realization

Assume that the symbol is given by a pair (ℓ, A) where $A^2 = 0$ and A has exactly one nonzero Jordan block of size 2. Let ℓ has signature (p, q).

Let $n = \frac{1}{2}(\dim M - 1)$. Then in coordinates $(z_1, \ldots, z_n, w)$ for $\mathbb{C}^{n+1}$ the (local) hypersurface realizations of the maximally symmetric models for this symbol are the hypersurfaces given by the equation

$$\mathrm{Im}(w + z_1^2 \bar{z}_n) = z_1 \bar{z}_2 + \bar{z}_1 z_2 + \sum_{i=3}^{n-1} \varepsilon_i z_i \bar{z}_i,$$

where $\varepsilon_i \in \{-1, 1\}$ and $\{\varepsilon_i\}_{i=3}^{n-1}$ consists of $p - 1$ terms equal to 1 and $q - 1$ terms equal to -1.

If $\dim M = 7$, then $n = 3$ and the model is:

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If $\dim M = 7$, then $n = 3$ and the model is:

$$\operatorname{Im}(w + z_1^2 \bar{z}_3) = z_1 \bar{z}_2 + \bar{z}_1 z_2.$$

Theorem (David Sykes and I. Z.)

Among all homogeneous 2-nondegenerate CR structures with one dimensional kernel the models with the symmetry algebra of maximal dimension are the models of the previous slide (i.e. the flat model for the nilpotent CR symbol with $A^2 = 0$ and A having exactly one nonzero Jordan block of size 2).

- A. Isaev, D. Zaitsev, *Reduction of five-dimensional uniformly Levi degenerate CR structures to absolute parallelisms.* J. Geom. Anal. 23 (2013), no. 3, 1571–1605.
- C. Medori , A. Spiro, *The equivalence problem for 5-dimensional Levi degenerate CR manifolds*, Int. Math. Res. Not. IMRN 2014, no. 20, 5602–5647.
- C. Porter, *The Local Equivalence Problem for 7-Dimensional, 2-Nondegenerate CR Manifolds whose Cubic Form is of Conformal Unitary Type*, preprint, arXiv:1511.04019v3 [math.DG].
- C. Porter and I. Zelenko, *Absolute parallelism for 2-nondegenerate CR structures via bigraded Tanaka prolongation*, preprint, arXiv:1704.0399v3[math.DG], 42 pages.
- D. Sykes and I. Zelenko, *A canonical form for pairs consisting of a Hermitian form and a self-adjoint antilinear operator*, Linear Algebra Appl. 590 (2020), 32?61

THANK YOU VERY MUCH FOR YOUR ATTENTION!