Homework Assignment 11 in MATH309-Spring 2013, ©Igor Zelenko

due April 24, 2013 . Show your work in all exercises

Topics covered: Elements of Fourier series and their applications in the method of separation of variables for the heat equation and Laplace equations on a rectangle

1. Graph each of the following functions and find the corresponding Fourier series

   (a)
   $$f(x) = \begin{cases} 2x, & 0 \le x \le 3 \\ 0, & -3 < x < 0 \end{cases}, \qquad \text{Period } 6;$$

   (b)
   $$f(x) = \begin{cases} 2-x, & 0 \le x \le 4 \\ x+2, & -4 < x < 0 \end{cases}, \qquad \text{Period } 8.$$

2. (a) Expand $f(x) = \cos x$, $0 < x < \pi$ in a <u>Fourier sine series</u>.

   (b) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \le x \le \pi$?

3. (a) Prove that for $0 \le x \le \pi$
   $$x(\pi - x) = \frac{\pi^2}{6} - \Big(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots\Big);$$

   (b) Prove that for $0 \le x \le \pi$
   $$x(\pi - x) = \frac{8}{\pi}\Big(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\Big).$$

   (c) Using item (a) show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Hint: Plug in a suitable value of x into the formula of item (a). Note that we already proved this identity in class using Fourier series of another function and Parseval's identity but here I want that you will not rely on what we did in class but on item (a) above only)

   (d) Using item (a) and Parseval's identity show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

   (e) (**bonus 10 pts**) Using item (b) and Parseval's identity show that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

4. Solve the initial-boundary value problem for the heat equation
   $$u_t = 4u_{xx}, \quad 0 < x < \pi, \ t > 0$$
   with the following initial and boundary conditions

   (a) $u(x,0) = x(\pi - x), \quad u(0,t) = u(\pi,t) = 0;$

   (b) $u(x,0) = x(\pi - x), \quad u_x(0,t) = u_x(\pi,t) = 0.$

   (Hint: In both items you can use first two items of Problem 3 above)

5. Solve the boundary value problem for Laplace equation
   $$u_{xx} + u_{yy} = 0$$
   on the rectangle $0 < x < \pi$, $0 < y < H$ with boundary conditions
   $$u(x,0) = 0, \ u(x,H) = \cos x, \ u(0,y) = u(\pi,y) = 0.$$

   (Hint: You can use the results of your calculations of Problem 2 above)