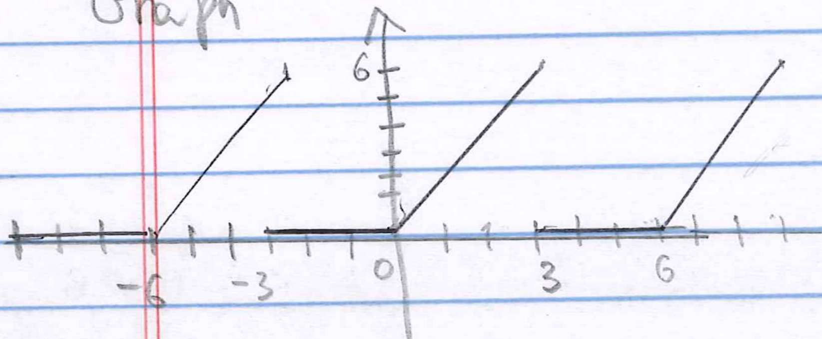
Homework #11 Solutions MATH309 Spring 2013

Problem 1

a) Graph



Find the coefficients of the Fourier series

$L = 3$

For $n \geq 1$ $a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{\pi n x}{3}\,dx = \frac{1}{3}\int_0^3 2x\cos\frac{\pi n x}{3}\,dx =$ integration by parts

$$= \underbrace{\frac{2}{3}\, x\, \frac{\sin\frac{\pi n x}{3}}{\frac{\pi n}{3}}\Big|_0^3}_{0} - \frac{2}{3\frac{\pi n}{3}}\int_0^3 \sin\frac{\pi n x}{3}\,dx =$$

$$= \frac{2}{\frac{\pi^2 n^2}{3}}\cos\frac{\pi n x}{3}\Big|_0^3 = \boxed{\frac{6}{\pi^2 n^2}(\cos \pi n - 1)} = \frac{6}{\pi^2 n^2}((-1)^n - 1)$$

$$= \begin{cases} 0 & n \text{ is odd} \\ -\frac{12}{\pi^2 n^2} & n \text{ is even} \end{cases}$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_0^3 2x\,dx = \frac{\cancel{2}}{3}\cdot\frac{x^2}{\cancel{2}}\Big|_0^3 = 3$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{\pi n x}{3}\,dx = \frac{1}{3}\int_0^3 2x\sin\frac{\pi n x}{3} =$$

$$= -\frac{2}{3}\left.\frac{x\cos\frac{\pi n x}{3}}{\frac{\pi n}{3}}\right|_0^3 + \frac{2}{3\frac{\pi n}{3}}\underbrace{\int_0^3 \cos\frac{\pi n x}{3}}_{0} =$$

$$= -\frac{2}{\pi n}\,3\cos\pi n = \boxed{-\frac{6}{\pi n}\cos\pi n} = -6\frac{(-1)^n}{\pi n}$$

The Fourier series is

$$\boxed{\frac{3}{2} + \sum_{n=1}^{\infty}\left(\frac{6}{\pi^2 n^2}(\cos\pi n - 1)\cos\frac{\pi n x}{3} - \frac{6}{\pi n}\cos\pi n\sin\frac{\pi n x}{3}\right)} =$$

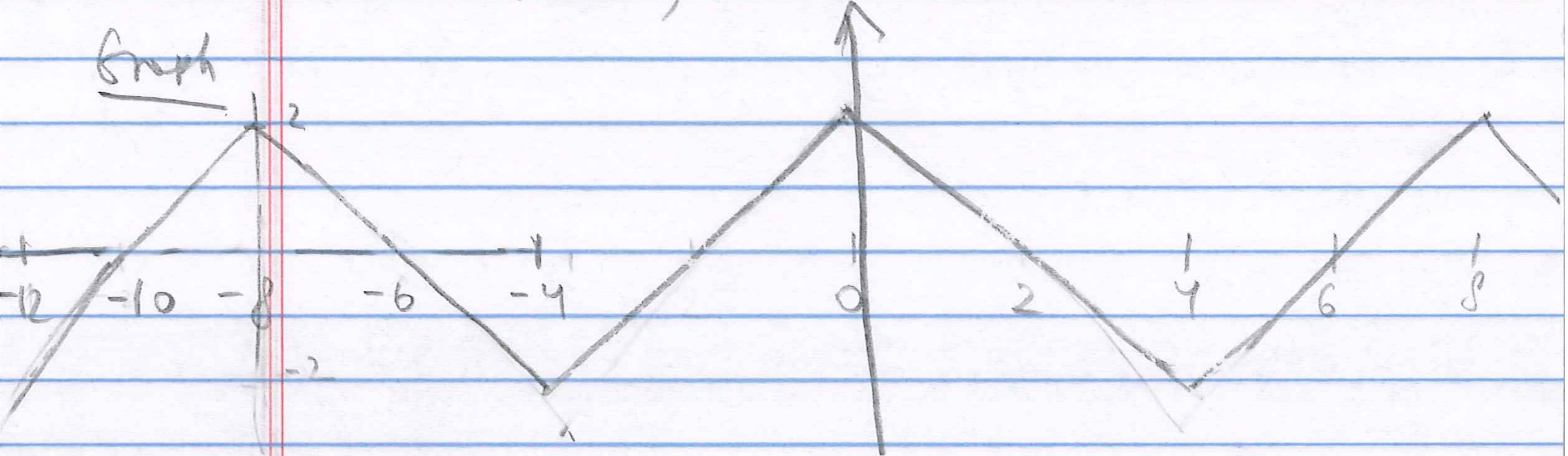
$$= \frac{3}{2} - \frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{\pi(2k-1)x}{3}}{(2k-1)^2} - \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{\pi n x}{3} =$$

Substitution: $n = 2k-1$

All possible, equivalent answers

(b) $f(x) = \begin{cases} 2-x, & 0 \le x \le 4 \\ x+2, & -4 < x < 0 \end{cases}$ period 8

Graph



This function is even $\Rightarrow b_n = 0$

$\underline{L=4}$

$n \ge 1 \quad a_n = \frac{1}{4} \int_{-4}^{4} f(x) \cos \frac{\pi n x}{4} \, dx = \frac{2}{4} \int_0^4 f(x) \cos \frac{\pi n x}{4} \, dx =$

$= \frac{1}{2} \int_0^4 (2-x) \cos \frac{\pi n x}{4} \, dx = \underbrace{\frac{(2-x)}{2} \frac{\sin \frac{\pi n x}{4}}{\frac{\pi n}{4}} \Big|_0^4}_{0} +$

$+ \frac{2}{\pi n} \int_0^4 \sin \frac{\pi n x}{4} \, dx = -\frac{2}{\frac{\pi^2 n^2}{4}} \cos \frac{\pi n x}{4} \Big|_0^4 = \frac{8}{\pi^2 n^2} (1 - \cos \pi n)$

$\frac{8}{\pi^2 n^2} (1-(-1)^n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{16}{\pi^2 n^2} & \text{if } n \text{ is odd} \end{cases}$

$a_0 = \frac{2}{4} \int_0^4 (2-x) \, dx = \frac{1}{2} \; 2x - \frac{x^2}{2} \Big|_0^4 = \frac{1}{2} \left(8 - \frac{16}{2}\right) = 0$

Therefore the Fourier series is

$$\boxed{\frac{16}{\pi^2}\left(\frac{\cos\frac{\pi}{4}x}{1^2}+\frac{\cos\frac{3\pi}{4}x}{3^2}+\frac{\cos\frac{5\pi}{4}x}{5^2}+\dots,\right)=\frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi}{4}x}{(2k-1)^2}}$$

Problem 2

(a) $f(x)=\cos x$, $0<x<\pi$ in Fourier sine

$L=\pi$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,dx = (*)$$

We use the following trig. formula

$\sin\alpha\cos\beta=\frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$. Take $\alpha=nx$, $\beta=x$

$$(*)=\frac{2}{\pi}\,\frac{1}{2}\int_0^{\pi}\left(\sin(n+1)x+\sin(n-1)x\right)dx \underset{\text{if } n\neq 1}{=}$$

$$=-\frac{1}{\pi}\left(\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right)\Bigg|_{x=0}^{\pi}=\frac{1}{\pi}\left(\frac{1-\cos(n+1)\pi}{n+1}+\right.$$

$$\left.+\frac{1-\cos(n-1)\pi}{n-1}\right)=\begin{cases}0, & \text{if } n \text{ is odd}\\ \frac{2}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right) & \text{if } n \text{ is even}\end{cases}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{8k}{\pi(4k^2-1)} & \text{if } n=2k \end{cases}$$

(I just plug in $n=2k$

If $n=1 \Rightarrow$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = 0$$

(recall that $\sin 2x = 2\sin x \cos x$)

So the required Fourier sine series

is $\boxed{\dfrac{8}{\pi}\displaystyle\sum_{k=1}^{\infty} \frac{k\sin(2kx)}{4k^2-1}}$ $(**)$

(b) If $x=0$, $x=\pi$ then $\sin 2kx = 0 \Rightarrow$ the series

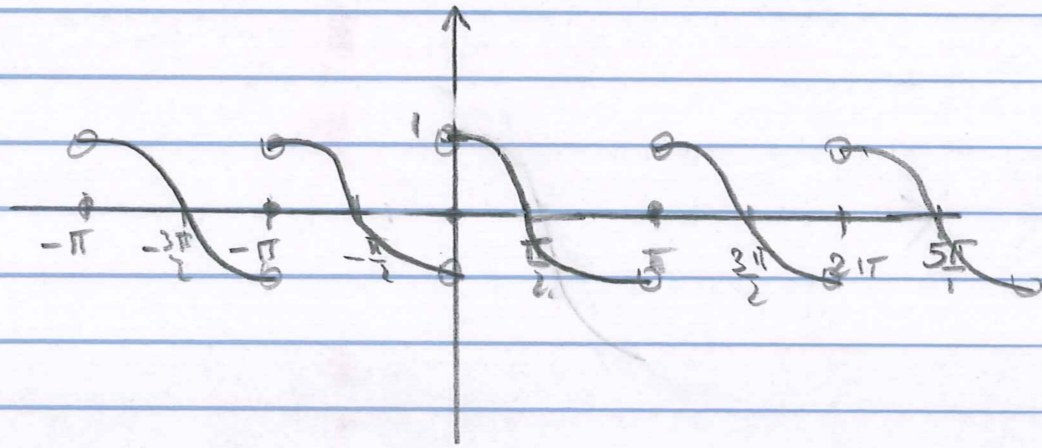
converge to $0 \Rightarrow$ we should make $f(0)=f(\pi)=0$

(Other explanation is using the Dirichlet thm

The Fourier sine series converge to an odd function

with period 2π s.t. it is equal to $\cos x$ on $0<x<\pi$

The graph of this function is



and the Fourier series at the points of discontinuity

converges to $\frac{f(x-)+f(x+)}{2}$

For $x=0$ $f(0-)=-1$, $f(0+)=1 \Rightarrow$ the Fourier

series converges to $\frac{-1+1}{2}=0 \Rightarrow$ We should set $f(0)=0$

In the same way $f(\pi)=0$)

Problem 3 a) We want to expand $f(x)=x(\pi-x)$, $0\le x\le\pi$

to the Fourier cosine series

$$a_0=\frac{2}{\pi}\int_0^{\pi}\underbrace{x(\pi-x)}_{\pi x-x^2}\,dx=\frac{2}{\pi}\left(\frac{\pi x^2}{2}-\frac{x^3}{3}\right)\Big|_0^{\pi}=\frac{2}{\pi}\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)=$$

$$=\frac{\pi^2}{3}\Rightarrow\frac{a_0}{2}=\frac{\pi^2}{6}$$

$L=\pi \Rightarrow \frac{\pi n x}{L} = nx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx \, dx = \underbrace{\frac{2}{\pi}\frac{(\pi x - x^2)\sin nx}{n}\Big|_0^{\pi}}_{0} -$$

$$- \frac{2}{\pi n}\int_0^{\pi} (\pi - 2x)\sin nx \, dx = \frac{2}{\pi n^2}(\pi - 2x)\cos nx \Big|_0^{\pi} +$$

$$+ \underbrace{\frac{4}{\pi n^2}\int_0^{\pi} \cos nx \, dx}_{0} = \frac{2}{\pi n^2}\Big((\pi - 2\pi)\cos \pi n - \pi\Big) =$$

$$= -\frac{2\pi}{\pi n^2}(\cos \pi n + 1) = -\frac{2}{n^2}((-1)^n + 1) = \begin{cases} -\frac{4}{n^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} =$$

$$= \begin{cases} -\frac{1}{k^2} & \text{if } n = 2k \\ 0 & \end{cases}$$

substitute $n=2k$

$\Rightarrow$ The Fourier cosine series is

$$\frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos 2kx}{k^2}$$

Note that if we extend $x(\pi - x)$ evenly from $(0,\pi)$ to $(-\pi,\pi)$ we and then to whole $\mathbb{R}$ as periodic of period 2π

we still get a continuous function with piecewise continuous derivatives (the jumps of the derivatives are at $x = \pi n$) $\Rightarrow$ by the Dirichlet Thm the Fourier cosine series converge to this periodic function $\Rightarrow$ we proved that

$$f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(2k x)}{k^2} = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots\right)$$

q.e.d.

b) We want to expand $f(x) = x(\pi - x)$, $0 \le x \le \pi$ in the Fourier sine series

$L = \pi \Rightarrow \frac{\pi n x}{L} = nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \underbrace{x(\pi - x)}_{\pi x - x^2} \sin nx \, dx = -\frac{2}{\pi} \frac{(\pi x - x^2)\cos nx}{n} \Big|_0^{\pi} +$$

$$+ \frac{2}{\pi n} \int_0^{\pi} (\pi - 2x) \cos nx \, dx = \underbrace{-\frac{2}{\pi} \frac{(\pi^2 - \pi^2)\cos n\pi - 0}{n}}_{0} +$$

$$+ \underbrace{\frac{2}{\pi n^2} (\pi - 2x) \sin nx \Big|_0^{\pi}}_{0} - \frac{2}{\pi n^2} \int_0^{\pi} (-2) \sin nx \, dx =$$

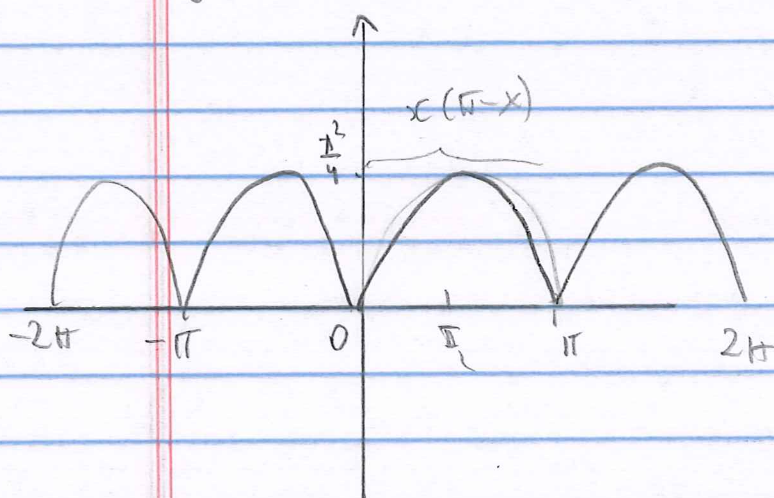
$$= -\frac{4}{\pi n^3} \cos nx \Big|_0^{\pi} = -\frac{4}{\pi n^3} (\cos \pi n - 1) = \begin{cases} 0 & n \text{ is even} \\ \frac{8}{\pi n^3} & n \text{ is odd} \end{cases}$$

Hence, the Fourier sine series of $x(\pi - x)$ on $(0,\pi)$ is

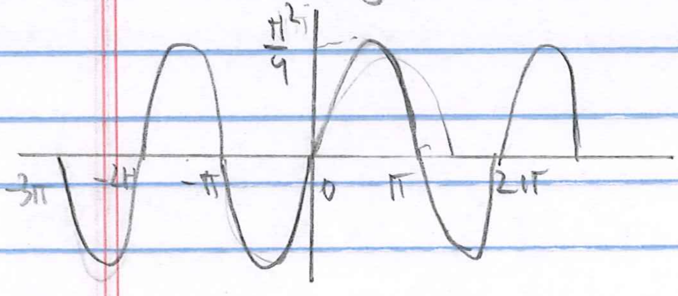
$$\frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right) \text{ and}$$

It converges to our function $x(\pi-x)$ by the same arguments as in the end of solution of the previous item.

Rem Note that in item a) the Fourier cosine series converges to the function with the graph



In the item b) the Fourier sine series converges to the function with the graph



(c) Plug $x=0$ into the formula

$$x(\pi-x)=\frac{\pi^2}{6}-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\dots\right)$$

$\Downarrow$

$$0=\frac{\pi^2}{6}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) \quad (\text{because } \cos 0=1)$$

$\Downarrow$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

(d) Let us use the Fourier cosine series of item (a)

Note that we have to use Parseval here for the even extension of $x(\pi-x)$ from the interval $(0,\pi)$ to the interval $(-\pi,\pi)$

$\Downarrow$

$$\underbrace{\frac{2}{\pi}\int_0^{\pi}(x(\pi-x))^2dx}_{\substack{\text{the integral of the square of the even extension over } (-\pi,\pi) = \\ = 2\times \text{ the integral of the original function over}}} = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_{2n})^2$$

(note that b_n here is equal to 0 and a_n with odd n are zeros)

On one hand, $\frac{2}{\pi}\int_0^{\pi}(x(\pi-x))^2dx=\frac{2}{\pi}\int_0^{\pi}x^2(\pi^2-2\pi x+x^2)dx=\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2-2\pi x^3+x^4)dx=$

$$=\frac{2}{\pi}\left(\frac{\pi^2x^3}{3}-\frac{\pi x^4}{2}+\frac{x^5}{5}\right)\Bigg|_0^{\pi}=2\pi^4\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=2\pi^4\left(-\frac{1}{6}+\frac{1}{5}\right)=\frac{2\pi^4}{30}=$$

$$=\frac{\pi^4}{15}$$

On the other hand $a_0=\frac{\pi^2}{3} \Rightarrow$ and $a_{2n}^2=\frac{1}{n^4} \Rightarrow$

$a_0^2=\frac{\pi^4}{9}$

$$\frac{\pi^4}{15}=\frac{\pi^4}{18}+\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{15}-\frac{\pi^4}{18}=\frac{\pi^4}{90} \text{ q.e.d.}$$

(e) Let us use the Fourier Sine series of Item 16)

Note that we have to use Parseval here for the odd extension of $x(\pi-x)$ from the interval $(0,\pi)$ to the interval $(-\pi,\pi)$

$$\underbrace{\frac{2}{\pi}\int_0^{\pi}\left(x(\pi-x)\right)^2 dx}_{\text{the same as in the previous item, because the square of the odd extension is even}} = \sum_{n=1}^{\infty}\left(b_{2n-1}\right)^2 \quad \left(\begin{array}{l}\text{because } a_n = 0 \\ \text{and } b_{2m} = 0\end{array}\right.$$

$$b_{2n-1} = \frac{8}{\pi(2n-1)^3} \Rightarrow b_{2n-1}^2 = \frac{64}{\pi^2}\frac{1}{(2n-1)^6} \Rightarrow$$

The ~~left~~ handside is already calculated in the previous item and is equal to $\frac{\pi^4}{15}$ so

$$\frac{\pi^4}{15} = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$$

But we need to calculated $\sum_{n=1}^{\infty}\frac{1}{n^6}$

let $S = \sum_{n=1}^{\infty}\frac{1}{n^6}$ then

$$S = \underbrace{\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\cdots\right)}_{\text{odd only}} + \underbrace{\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\cdots\right)}_{\text{even only}} =$$

$$= \underbrace{\frac{\pi^6}{960}}_{\text{by above}} + \frac{1}{2^6}\underbrace{\left(\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\cdots\right)}_{S} \Rightarrow S = \frac{\pi^6}{960} + \frac{S}{64} \Rightarrow$$

$$\frac{63}{64}S = \frac{\pi^6}{960} \Rightarrow S = \frac{64}{63}\frac{\pi^6}{960} = \frac{\pi^6}{63\cdot 15} = \frac{\pi^6}{945} \text{ q.e.d.}$$

Problem 4 (a) Solve the initial/boundary value problem

$$u_t = 4u_{xx} \quad 0<x<\pi, \; t>0$$

$$u(x,0) = x(\pi - x), \quad u(0,t) = u(\pi,t)$$

Solution According to the algorithm given to such kind of initial/boundary value problems in lecture 6, 04/22, first we have to expand the initial function $x(\pi-x)$ into the Fourier sine series. This was already done in Problem 3 b)

$$x(\pi - x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$$

$\Downarrow$ (see formula (6), p. 2, lecture 6, 04/22) with $k=4$, $L=\pi$

$$\boxed{u(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \underbrace{\frac{1}{(2n-1)^3}}_{b_{2n-1}} e^{-4(2n-1)^2 t} \sin(2n-1)x}$$

(note that only b_n with odd n are not zero), this is the reason

(b) $u_t = 4u_{xx}$

$$u(x,0) = x(\pi - x), \quad u_x(0,t) = u_x(\pi,t)$$

Solution According to the algorithm given to such kind of initial/boundary value problems in lecture 6, 04/22, first we have to expand the initial function $x(\pi - x)$ into Fourier cosine series. This was already done

In problem 3(a)

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^2}$$

$\Downarrow$ by formula 19) p.7 lecture 6, 04/22 with $k=4$, $L=\pi$

$$u(x,t) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{a_{2n}} e^{-4\cdot 4n^2 t} \cos 2nx =$$

$$= \boxed{\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{e^{-16n^2 t}}{n^2} \cos 2nx}$$

Problem 5 Solve

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < \pi, \ 0 < y < 4$$

$$u(x,0) = 0, \ u(x,L) = \cos x, \ u(0,y) = u(\pi, y) = 0$$

Solution According to the algorithm described in lecture notes of Lecture 6, p. 10-15 first we have to find the Fourier sine series of the function on the top edge of the boundary. We already calculated it in Problem 2a

$$\cos x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin(2nx) \Rightarrow$$

$\Rightarrow$ using formulas (18) & (19) of lec 6 p.14

(with $L = H$)

$$\boxed{u(x,y) = \frac{8}{\pi} \sum_{n=1}^{\infty} \underbrace{\left(\frac{n}{(4n^2-1)\sinh 2nH} \right)}_{b_{2n}} \sinh(2ny) \sin(2nx)}$$

(note that $b_{2n-1} = 0$)