Homework #12 MATH 309 Solutions

Problem 1 Solve the following one-dimensional wave equation

$$u_{tt} = 4u_{xx}, \quad 0<x<3, \quad t>0$$

with the initial conditions

$$u(x,0) = x^2, \quad u_t(x,0) = 0$$

and the boundary conditions

$$u(0,t) = u(3,t) = 0, \quad t>0$$

Solution First find solutions in the form

$$u(x,t) = X(x)T(t)$$

Substituting into equation:

$$XT'' = 4X''T \Rightarrow \quad \frac{X''}{X} = \frac{T''}{4T} = c \Rightarrow$$

$$X'' - cX = 0$$

$$T'' - 4cT = 0$$

$u(0,t) = u(3,T) = 0 \quad \Rightarrow X(0) = X(3) = 0 \Rightarrow$ by analogy with what we did previously for the heat equation and the Laplace equation $c = -\frac{\pi n^2}{L^2} = -\frac{\pi^2 n^2}{9}, n \in \mathbb{N}$ (here $L=3$) and

$$X(x) = B \sin \frac{\pi n}{3} x$$

Analyze now the equation for T.

$$T'' + \frac{4\pi^2 n^2}{9} T = 0 \Rightarrow T(t) = C \cos \frac{2\pi n}{3} t + D \sin \frac{2\pi n}{3} t$$

The condition $u_t(x,0) = 0$ implies that $T'(0) = 0 \Rightarrow$

$$\left. -C\frac{2\pi n}{3} \sin \frac{2\pi n}{3} t + \frac{2\pi n}{3} D \cos \frac{2\pi n}{3} t \right|_{t=0} = \frac{2\pi n}{3} D = 0 \Rightarrow$$

$P=0 \Rightarrow T(t) = C\cos\frac{2\pi n}{3}t \Rightarrow$

$$u(x,t) = b_n \cos\frac{2\pi n}{3}t \sin\frac{\pi n}{3}x$$

We seek for the solution satisfying $u(x,0)=x^2$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{2\pi n}{3}t \sin\frac{\pi n}{3}x \qquad (*)$$

So $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n}{3}x = x^2 \qquad 0<x<3$

Therefore to find coefficients b_n we have to expand x^2 in the <u>Fourier sine</u> series on the interval $(0,3)$

$$b_n = \frac{2}{3}\int_0^3 x^2 \sin\frac{\pi n}{3}x\,dx = \frac{2}{3}\cdot -\frac{x^2\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Bigg|_0^3 + \frac{2}{3}\int_0^3 \frac{2x\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}\,dx =$$

$$-\frac{2}{\pi n}x^2\cos\frac{\pi n}{3}x\Bigg|_0^3 + \frac{4}{\pi n}\int_0^3 x\cos\left(\frac{\pi n}{3}x\right)dx =$$

$$= -\frac{2}{\pi n}9\cos\pi n + \frac{4}{\pi n}\frac{x\sin\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Bigg|_0^3 - \frac{4}{\pi n}\int_0^3 \frac{\sin\frac{\pi n}{3}x}{\frac{\pi n}{3}}dx =$$

$$= -\frac{18}{\pi n}(-1)^n + \frac{12}{\pi^2 n^2}\frac{\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Bigg|_0^3 = \frac{18}{\pi n}(-1)^{n+1} + \frac{36}{\pi^3 n^3}(\cos\pi n - 1) =$$

$$= \underbrace{\frac{18}{\pi n}(-1)^{n+1} + \frac{36}{\pi^3 n^3}((-1)^n - 1)}_{=\frac{18}{\pi n}\left((-1)^{n+1} + \frac{2}{\pi^2 n^2}((-1)^n-1)\right)} = \begin{cases} \frac{18}{\pi n} - \frac{72}{\pi^3 n^3} & \text{if } n \text{ is odd} \\ -\frac{18}{\pi n} & \text{if } n \text{ is even} \end{cases}$$

substituting to $(*)$ $\Rightarrow$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{18}{\pi n}\left((-1)^{n+1} + \frac{2}{\pi^2 n^2}((-1)^n - 1)\right)\cos\frac{2\pi n}{3}t \sin\frac{\pi n}{3}x}$$

Problem 2 Solve the following boundary value

$$u_{xx}+u_{yy}=0 \text{ for } x^2+y^2<1$$

$$u(x,y)=y^3 \qquad x^2+y^2=1$$

Solution Passing to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$

we get

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$u(1,\theta)=\sin^3\theta$, $u(r,\theta)$ is periodic w.r.t. θ with period 2π, $u(0,\theta)$ is finite.

Seperate the variables $u(r,\theta)=R(r)\Theta(\theta)$.

Substituting into equation

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0 \Rightarrow \text{ dividing by } R\Theta$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=0 \Rightarrow$$

$$r^2\left(\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}\right)=-\frac{\Theta''}{\Theta}=C$$

$$\begin{cases} \dfrac{r^2R''+rR'}{R}=C & (1) \\ \Theta''+C\Theta=0 & (2) \end{cases}$$

$\Theta''+C\Theta=0$ with $\Theta(\theta)$ being periodic with period 2π

$\Rightarrow C=n^2$, $n\geq 0$, n is integer (see lecture #7, class of April 26 for more details on this) $\Rightarrow \Theta''+n^2\Theta=0 \Rightarrow$

$\Theta(\theta)=a_n\cos n\theta+b_n\sin n\theta$ if $n>0$ or $\Theta(\theta)\equiv \text{const}$ in $n=0$

Substituting $C=n^2$ into (1) we get the Euler equation

$$r^2 R'' + rR' - n^2 R = 0$$

and the gen. solution of it is

$$R(r) = cr^n + dr^{-n} \quad \text{if } n>0$$

$$R(r) = c + d\ln r \quad \text{if } n=0$$

Recall that $u(0,\theta)$ is finite $\Rightarrow$ $R(0)$ is finite $\Rightarrow$ in both cases

$$d = 0 \Rightarrow \quad R(r) = cr^n, \quad n \in \underbrace{\mathbb{N}\cup\{0\}}_{\text{natural numbers or } 0}$$

$\Downarrow$

Our "building blocks" i.e. solutions of the type $R(r)\Theta(\theta)$ which are periodic in θ and finite at $r=0$ are of the form

$$r^n\,(a_n \cos n\theta + b_n \sin n\theta) \quad (*)$$

Now we look for the solution in the form of infinite superposition of solutions of type $(*)$ i.e. in the form

$$u(r,\theta) = \underbrace{\frac{a_0}{2}}_{\text{constant}} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \quad (*)$$

such that $u(1,\theta) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \sin^3\theta$

It means that in order to find the appropriate coefficients a_n and b_n we have to express $\sin^3\theta$ in the Fourier series of period 2π

Note that $\sin^3\theta = \sin\theta \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} = \frac{1}{2}\sin\theta\,(1-\cos 2\theta) =$

$= \frac{1}{2}\sin\theta - \frac{1}{2}\underbrace{\sin\theta\cos 2\theta}_{\frac{1}{2}(\sin 3\theta - \sin\theta)}$ $= \frac{1}{2}\sin\theta - \frac{1}{4}\sin 3\theta + \frac{1}{4}\sin\theta =$

$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \Rightarrow$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

This is exactly the Fourier series of $\sin^3\theta$ with period 2π ($\sin^3\theta$ is just a trigonometric polynomial, its Fourier series consist of 2 terms only)

$\Downarrow$

$a_n = 0$ and $b_1 = \frac{3}{4}$, $b_3 = -\frac{1}{4}$

$b_n = 0$ for all other n.

$\Downarrow$ substituting into $(*)$

$$\boxed{u(r,\theta) = \frac{3}{4} r \sin\theta - \frac{1}{4} r^3 \sin 3\theta}$$

If we want to express this in the original cartesian coordinates (preferable but not required)

$\sin 3\theta = \sin(2\theta + \theta) = \underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\cos\theta + \underbrace{\cos 2\theta}_{\cos^2\theta - \sin^2\theta}\sin\theta =$

$= 2\sin\theta\cos^2\theta + \cos^2\sin\theta - \sin^3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$u(r,\theta) = \frac{3}{4} r\sin\theta - \frac{1}{4} r^3 \sin 3\theta = \frac{3}{4} y - \frac{3}{4}\underbrace{r^2\cos^2\theta}_{x^2}\underbrace{r\sin\theta}_{y} + \frac{1}{4}\underbrace{r^3\sin^3\theta}_{y^3} =$

$$= \boxed{\frac{3}{4} y - \frac{3}{4} x^2 y + \frac{1}{4} y^3}$$