Homework assignment 1 in MATH 309
Solutions

1. (a) $2x_1 + x_2 = 6$ is a line passing through

$(3,0)$ and $(0,6)$ (this are intercepts with the axis

$x_1 - 3x_2 = -4$ is a line passing through

$(-4,0)$ and $(0, \frac{4}{3})$

This two lines are not parallel

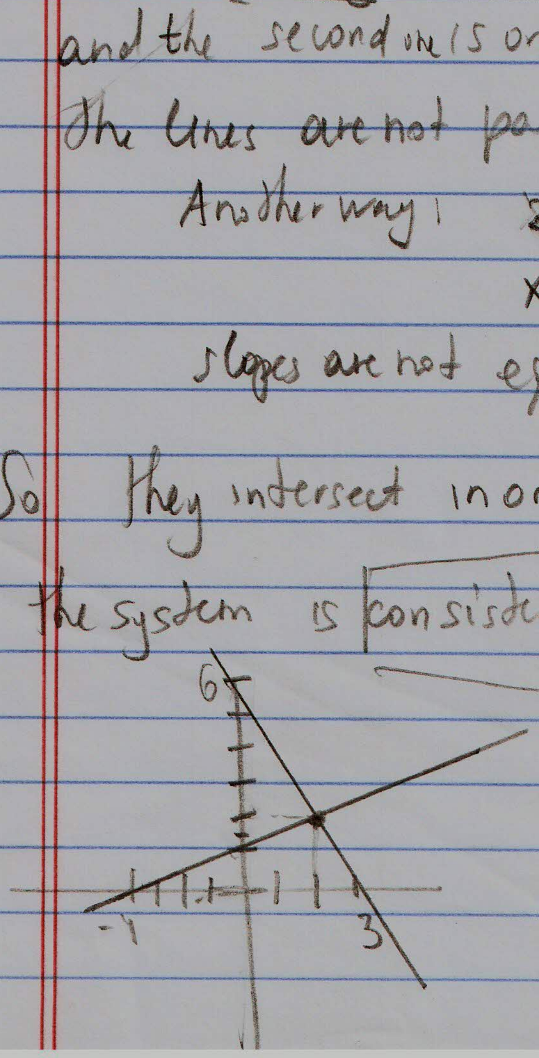
Explanation: <u>One way</u> The first line is orthogonal to the vector $(2,1)$ and the second one is ortogonal to $(1,-3)$; $(2,1) \nparallel (1,-3) \Rightarrow$ the lines are not parallel

Another way: $2x_1 + x_2 = 6 \Rightarrow x_2 = -2x_1 + 6 \Rightarrow$ slope $= -2$

$x_1 - 3x_2 = -4 \Rightarrow x_2 = \frac{1}{3}x_1 + \frac{4}{3} \Rightarrow$ slope $= \frac{1}{3}$

slopes are not equal $\Rightarrow$ the lines are not parallel

So they intersect in one point $\Rightarrow$ there is $\boxed{\text{one solution}} \Rightarrow$ the system is $\boxed{\text{consistent}}$



Algebraic solution

$2x_1 + x_2 = 6$

$x_1 - 3x_2 = -4$

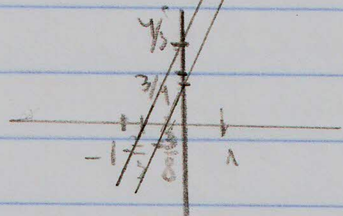
Eq 1 - 2 Eq 2 : $7x_2 = 14 \Rightarrow x_2 = 2 \Rightarrow$

$2x_1 + 2 = 6 \Rightarrow x_1 = 2 \Rightarrow$

Solution is $\boxed{(2,2)}$

(b) $\begin{cases} 6x_1 - 3x_2 = -4 \iff 2x_1 - x_2 = -\frac{4}{3} \iff x_2 = 2x_1 + \frac{4}{3} \\ -8x_1 + 4x_2 = 3 \iff 2x_1 - x_2 = -\frac{3}{4} \iff x_2 = 2x_1 + \frac{3}{4} \end{cases}$

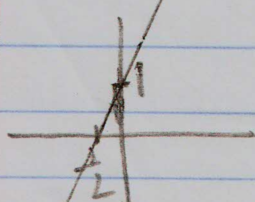
Two parallel lines (orthogonal to the vector $(2,-1)$) or with slope 2) which do not coincide) $\Rightarrow$ the system is $\boxed{\text{inconsistent}}$



(c) $\begin{cases} 6x_1 - 3x_2 = -3 \iff 2x_1 - x_2 = -1 \iff x_2 = 2x_1 + 1 \\ -8x_1 + 4x_2 = 4 \iff 2x_1 - x_2 = -1 \end{cases}$ $\rightarrow$ Two lines coinside $\Rightarrow$

$\boxed{\text{Consistent}}$ $\boxed{\text{Infinitely many solutions}}$

$\boxed{(\alpha, 2\alpha + 1)}$



Problem 2 From the last equation: $7x_5 = -7 \Rightarrow \boxed{x_5 = -1}$

Substitute x_5 to the Eq 4: $10x_4 - 5 = 5 \Rightarrow 10x_4 = 10 \Rightarrow \boxed{x_4 = 1}$

Substitute x_4 & x_5 into Eq 3: $3x_3 + 5 - 1 \Rightarrow x_3 = -\frac{4}{3}$

Substitute: x_3, x_4 & x_5 into Eq 2: $2x_2 + \frac{4}{3} - 2 + 4 = -4 \Rightarrow$

$$2x_2 = -6 - \frac{4}{3} = -\frac{22}{3} \Rightarrow x_2 = -\frac{11}{3}$$

Substitute: x_2, x_3, x_4 & x_5 into Eq 1

$$3x_1 - \frac{55}{3} - \frac{8}{3} - 4 + 1 = 4$$

$$3x_1 = \frac{63}{3} + 7 = 21 + 7 = 28 \Rightarrow x_1 = \frac{28}{3}$$

Answer: $\left(\frac{28}{3}, -\frac{11}{3}, -\frac{4}{3}, 1, -1\right)$

Problem 3 (Section 1.2, p.23, problem 1)

(a) It is in row echelon form but not in the reduced row echelon form, because above the leading coefficient of the second row there is a nonzero entry in the same column

(b) It is not in row echelon form (and therefore not in reduced row echelon form), because row ~~th~~three is nonzero row and its below the row two, which consists of zeros

(c) It is in row echelon and in the reduced row echelon forms

(d) It is in row echelon and in the reduced row echelon forms

(e) It is not in row echelon form (and therefore not in the reduced row echelon form), because the first nonzero entry of the third row is 3

(f) It is not in row echelon form (and therefore not in reduced row echelon form) because # of leading zeros in the 3rd row is 1 and less than # of leading zeros in the second column

(g) It is in row echelon and in the reduced row echelon form

(h) It is in row echelon form but not in the reduced row echelon form because the leading entry in the second row is not the only nonzero entry in its column (there is 3 above)

Problem 4

(a) $\boxed{\text{Inconsistent}}$: the leading entry of the third row is in the last column, so the third equation is $0 = 1 \Rightarrow$ inconsistent

(b) $\boxed{\text{Consistent}}$ and has $\boxed{\text{one solution}}$ (because there is no free variables) Find the solution by back substitution: Eq 3: $x_3 = 1 \Rightarrow$ Eq 2: $x_2 + 7 = 5 \Rightarrow x_2 = -2$ $\Rightarrow$ Eq 1: $x_1 - 10 + 3 = -6 \Rightarrow x_1 = 1 \Rightarrow$ The solution is

$$\boxed{(1, -2, 1)}$$

(c) It is $\boxed{\text{consistent}}$ and has infinite many solutions: as free variables one can take $\boxed{x_2 \ \& \ x_4}$

Problem 5 (a) Augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & 3 & -1 & -1 & 2 \\ 1 & 2 & 1 & -1 & 1 \\ 3 & 5 & 0 & 0 & 3 \\ 1 & 1 & -2 & 2 & 1 \end{array}\right)$$

Let us transform it to a row echelon form

$$\left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 2 \\ 1 & 2 & 1 & -1 & 1 \\ 3 & 5 & 0 & 0 & 3 \\ 1 & 1 & -2 & 2 & 1 \end{array}\right) \xrightarrow[\text{to avoid fractions}]{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 2 & 3 & -1 & 1 & 2 \\ 3 & 5 & 0 & 0 & 3 \\ 1 & 1 & -2 & 2 & 1 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 2R_1}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -1 & -3 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \to -R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -1 & -3 & 3 & 0 \end{array}\right) \xrightarrow[R_4 \to R_3 + R_2]{R_3 \to R_3 + R_2}$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ $\Rightarrow$ system is consistent, because there is no leading entry in the last row

Free variables x_3 and x_4 $\Rightarrow$ let $x_3 = \alpha$, $x_4 = \beta$

One way: back substitution:

$$x_1 + 2x_2 + \alpha - \beta = 1$$

$$x_2 + 3\alpha - 3\beta = 0 \Rightarrow x_2 = 3\beta - 3\alpha \Rightarrow \text{substitute}$$

in eq 1: $x_1 + 6(\beta - \alpha) + \alpha - \beta = 1 \Rightarrow x_1 = 1 - 6(\beta-\alpha) + \beta - \alpha =$

$= 1 - 5(\beta - \alpha) \Rightarrow$

Solution is $\boxed{1-5(\beta-\alpha),\ 3(\beta-\alpha),\ \alpha,\ \beta}$

Another way : reduced echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & -5 & 5 & 1 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $x_1 - 5\alpha + 5\beta = 1$ $\Rightarrow$ $x_1 = 1 - 5(\beta - \alpha)$

$x_2 + 3\alpha - 3\beta = 0$ $\quad x_2 = 3(\beta - \alpha)$

Problem 5 (b) Augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 1 \\ -3 & 10 & -5 & -1 & -1 \\ -2 & 7 & -2 & -1 & 5 \end{array}\right)$$

Transform it to row echelon form

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 1 \\ -3 & 10 & -5 & -1 & -1 \\ -2 & 7 & -2 & -1 & 5 \end{array}\right) \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right) \Rightarrow \boxed{\text{consistent}} \text{ and have}$$

infinite many solution

free variable : x_4

One way Back substitution: $x_4 = \alpha$

Eq 3: $x_3 - \alpha = 5 \Rightarrow \boxed{x_3 = 5 + \alpha}$

Eq 2: $x_2 + x_3 + 2\alpha = 2 \Rightarrow x_2 + 5 + \alpha + 2\alpha = 2 \Rightarrow \boxed{x_2 = -3 - 3\alpha}$

Eq 1: $x_1 - 3x_2 + 2x_3 + \alpha = 1 \Rightarrow x_1 - 3(-3-3\alpha) + 2(5+\alpha) + \alpha = 1$

$x_1 + 9 + 9\alpha + 10 + 2\alpha + \alpha = 1 \quad x_1 + 12\alpha = -18 \Rightarrow \boxed{x_1 = -18 - 12\alpha}$

$\Downarrow$

The set of solutions is

$$\boxed{(-18-12\alpha, -3-3\alpha, \alpha+5, \alpha) \quad \text{for arbitrary real } \alpha}$$

Another way - transform the row echelon form to the reduced row echelon

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right) \xrightarrow{R_1 \to R_1 + 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & 5 & 7 & 7 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right) \to$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 5R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 12 & -18 \\ 0 & 1 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right) \Rightarrow$$

$x_1 = -x_4 - 18$
$x_2 = -3x_4 - 3$ $\Rightarrow$ the same answer
$x_3 = x_4 + 5$

Problem 5 (c)

$$\begin{cases} x_1 + 2x_2 - x_3 = 6 \\ -x_1 + 2x_2 + 2x_3 = 5 \\ 4x_2 + x_3 = 10 \end{cases}$$

Augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ -1 & 2 & 2 & 5 \\ 0 & 4 & 1 & 10 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 4 & 1 & 11 \\ 0 & 4 & 1 & 10 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & -1 \end{array}\right) \to \boxed{\text{inconsistent}} - \text{there is leading entry in the last column}$$

$$\Big\downarrow \begin{array}{l} R_2 \to \frac{1}{4} R_2 \\ R_3 \to -R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1/4 & 11/4 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \text{row echelon form}$$

Problem 6 (Bonus)

Transform to the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 5 & 4 & 6 \\ 2 & 9 & 1 & 2 \\ 1 & 2 & a & b \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 5 & 4 & 6 \\ 0 & -1 & -7 & -10 \\ 0 & -3 & a-4 & b-6 \end{array}\right) \to$$

$$R_2 \to -R_2 \quad \left(\begin{array}{ccc|c} 1 & 5 & 4 & 6 \\ 0 & 1 & 7 & 10 \\ 0 & -3 & a-4 & b-6 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_1} \left(\begin{array}{ccc|c} 1 & 5 & 4 & 6 \\ 0 & 1 & 7 & 10 \\ 0 & 0 & \underbrace{a-4+21}_{a+17} & \underbrace{b-6+30}_{b+24} \end{array}\right)$$

a) The system has infinitely many solution if

$$\boxed{a = -17 \text{ and } b = -24}$$

(If $a \neq -17$ the system has a unique solution

If $a = -17$ but $b \neq -24$ the system is inconsistent

b) $\boxed{a = -17 \text{ and } b \neq -24}$ the