

Homework assignment 2 in MATH309 - Spring 2013 Solutions

Problem 1

$$A = \begin{pmatrix} 2 & -1 & 5 \\ -3 & 4 & 1 \\ 6 & -5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 3 & -4 \\ 4 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

$$(a) \quad 3A - BA = \begin{pmatrix} 6 & -3 & 15 \\ -9 & 12 & 3 \\ 18 & -15 & -12 \end{pmatrix} - \begin{pmatrix} -2 & 3 & -4 \\ 4 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ -3 & 4 & 1 \\ 6 & -5 & -4 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & -3 & 15 \\ -9 & 12 & 3 \\ 18 & -15 & -12 \end{pmatrix} - \begin{pmatrix} -4-9-24 & 2+12+20 & -10+3+16 \\ -8+9+6 & -4-12+5 & 20-3+4 \\ 4+3-18 & -2-4+15 & 10-1+12 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & -3 & 15 \\ -9 & 12 & 3 \\ 18 & -15 & -12 \end{pmatrix} - \underbrace{\begin{pmatrix} -37 & 34 & 9 \\ 11 & -11 & 21 \\ -11 & 9 & 21 \end{pmatrix}}_{BA} = \begin{pmatrix} 43 & -37 & 6 \\ -20 & 23 & -18 \\ 29 & -24 & -33 \end{pmatrix}$$

$$(b) \quad A^T B^T = (BA)^T = \begin{pmatrix} -37 & 11 & -11 \\ 34 & -11 & 9 \\ 9 & 21 & 21 \end{pmatrix}$$

from the
calculation
of BA in
the previous
item

$$(c) \quad (BA)^T = A^T B^T = \text{the same answer as in (b)}$$

(a) Both AB and BA make sense

$$AB = \begin{pmatrix} 1 & 5 & -4 \\ 2 & -7 & 8 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -2 & 1 \\ 3 & -4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 4 + 5 \cdot (-2) + (-4) \cdot 3 & 1 \cdot (-3) + 5 \cdot 1 + (-4) \cdot (-4) \\ 2 \cdot 4 + (-7) \cdot (-2) + 8 \cdot 3 & 2 \cdot (-3) + (-7) \cdot 1 + 8 \cdot (-4) \end{pmatrix} =$$

$$= \begin{pmatrix} 4 - 10 - 12 & -3 + 5 + 16 \\ 8 + 14 + 24 & -6 - 7 - 32 \end{pmatrix} = \begin{pmatrix} -18 & 18 \\ 46 & -45 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & -3 \\ -2 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 5 & -4 \\ 2 & -7 & 8 \end{pmatrix} = \begin{pmatrix} 4 - 6 & 20 + 21 & -16 - 24 \\ -2 + 2 & -10 - 7 & 8 + 8 \\ 3 - 8 & 15 + 28 & -12 - 32 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & 41 & -40 \\ 0 & -17 & 16 \\ -5 & 43 & -44 \end{pmatrix}$$

(b) AB does not make sense because $\# \text{ columns of } A = 3 \neq$
 $\# \text{ rows of } B = 2$

BA makes sense:

$$BA = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & -4 \\ 2 & -7 & 8 \end{pmatrix} = \begin{pmatrix} -2 & 41 & -40 \\ 0 & -17 & 16 \end{pmatrix}$$

The first two rows of
the BA from the previous item

(c) Both AB and BA make sense

$$AB = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} \begin{pmatrix} -3 & 10 & -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-3) & 2 \cdot 10 & 2 \cdot (-1) \\ (-1) \cdot (-3) & (-1) \cdot 10 & (-1) \cdot (-1) \\ (-4) \cdot (-3) & (-4) \cdot 10 & (-4) \cdot (-1) \end{pmatrix} =$$

$$= \begin{pmatrix} -6 & 20 & -2 \\ 3 & -10 & 1 \\ 12 & -40 & 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 10 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = -3 \cdot 2 + 10 \cdot (-1) + (-1) \cdot (-4) =$$

$$= -6 - 10 + 4 = -12$$

Problem 3 $x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \Rightarrow$

$$\begin{aligned} x_1 + 4x_2 &= 3 \\ -2x_1 + 5x_2 &= -4 \end{aligned} \xrightarrow{2 \text{Eq1} + \text{Eq2}}$$

$$8x_2 + 5x_2 = 6 - 4 \Rightarrow 13x_2 = 2 \Rightarrow$$

$$\boxed{x_2 = \frac{2}{13}} \Rightarrow x_1 = 3 - 4 \cdot \frac{2}{13} = 3 - \frac{8}{13} = \frac{31}{13}$$

$$\Rightarrow \boxed{b = \frac{31}{13} a_1 + \frac{2}{13} a_2}$$

Problem 4

First of all the system $Ax=b$ is consistent because from the fact that

$$b = 3a_1 - 4a_2 + 5a_3 - 6a_4 + 7a_5$$

it follows that $A \begin{pmatrix} 3 \\ -4 \\ 5 \\ -6 \\ 7 \end{pmatrix} = b$, i.e.

$x = \begin{pmatrix} 3 \\ -4 \\ 5 \\ -6 \\ 7 \end{pmatrix}$ is a solution of $Ax=b$.

Second, the system is underdetermined

(# of eq = 3 and # of variables = 5, i.e. # eq < # variables)

\Rightarrow there are at least $5-3=2$ free variables

(in a row echelon form) \Rightarrow since $Ax=b$ is consistent it has infinite many solutions

Problem 3 of section 1.4, p. 56

Find nonzero 2×2 matrices A and B such that $AB=0$

Example Take $A=B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\text{Then } AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Problem 4 of section 1.4 p. 56

Find nonzero matrices A, B , and C such that

$$AC = BC$$

Example Take, for example, $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$B = C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Then } AC = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$BC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow AC = BC$$

Problem numbered as #6 in the HWK assignment

$$(a) \quad A = \begin{pmatrix} 9 & -8 \\ -10 & 9 \end{pmatrix}$$

$$\det A = 9 \cdot 9 - (-8) \cdot (-10) = 81 - 80 = 1$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 9 & 8 \\ 10 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 10 & 9 \end{pmatrix}$$

$$(b) \quad x = A^{-1} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 10 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 27 - 16 \\ 30 - 18 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix}$$

Problem 14 Section 1.4 p.56

Prove by contradiction: Assume that A is nonsingular.

Then $AB = A \Rightarrow A^{-1}(AB) = A^{-1}A \Rightarrow$
associativity

$$\underbrace{(A^{-1}A)}_I B = I \Rightarrow B = I, \text{ but it is given that } B \neq I \Rightarrow$$

contradiction $\Rightarrow A$ must be singular

Problem 15 Section 1.4 p.56

A is nonsingular \Rightarrow there exists the inverse matrix A^{-1}
 such that

$$A^{-1}A = AA^{-1} = I$$

but the same identity means that A is the
 inverse of A^{-1} , i.e. $(A^{-1})^{-1} = A$.

Problem 16 Section 1.4 p.56

To prove that $(A^T)^{-1} = (A^{-1})^T$ we have to show
 that $(A^{-1})^T A^T = A^T (A^{-1})^T = I$:

$$(A^{-1})^T A^T = (A A^{-1})^T = I^T = I$$

\downarrow
 property (4) of transpose

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

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Problem 21, section 1.4 p.57

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

One way R is nonsingular $\Rightarrow \det R \neq 0$

$$\det R = \cos^2 \theta + \sin^2 \theta = 1 \neq 0$$

By exercise 12 p.56 (proved also in class)

$$R^{-1} = \frac{1}{\det R} \underbrace{\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}}_{R^T} = R^T$$

Another way let us check that $R^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

is indeed the inverse of R by checking that

$$R R^T = R^T R = I:$$

$$\begin{aligned} R R^T &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$R^T R = I$ in a similar way (actually, it is enough to check $R R^T = I$ as we discussed)