Homework assignment #3, Solutions, MATH309

Problem 1 (a)

$$\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{first row}}}{=} -a \begin{vmatrix} b & d \\ 0 & 0 \end{vmatrix} = \boxed{0}$$

(b)

$$\begin{vmatrix} -10 & 4 & -1 & 6 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 7 & -3 & 2 & 8 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{second row}}}{=} 3 \begin{vmatrix} -10 & -1 & 6 \\ 0 & 0 & -2 \\ 7 & 2 & 8 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{second row}}}{=} 3\cdot(-(-2)) \begin{vmatrix} -10 & -1 \\ 7 & 2 \end{vmatrix} =$$

$$= 6\,(-20+7) = -6\cdot 13 = \boxed{-78}$$

(c)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \overset{\substack{R_2\to R_2-R_1\\ R_3\to R_3-R_2}}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0 \text{ (because 2 rows coinside}$$

Problem 2 (a)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -4 & 0 & 1 & 0 \\ 7 & 6 & -5 & 0 & 0 & 1 \end{array}\right) \underset{R_3\to R_3-7R_1}{\overset{R_2\to R_2-3R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & -8 & 2 & -7 & 0 & 1 \end{array}\right) \overset{R_3\to R_3+8R_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & -6 & -31 & 8 & 1 \end{array}\right) \underset{R_3\to -\frac{1}{6}R_3}{\overset{R_1\to R_1-2R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 7 & -2 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right) \underset{R_2\to R_2+R_3}{\overset{R_1\to R_1-R_3}{\sim}}$$

The matrix A is invertible; Because there is no free variables

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7-\frac{31}{6} & -2+\frac{4}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -3+\frac{31}{6} & 1-\frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{6} & -\frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{13}{6} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{11}{6} & -\frac{2}{3} & \frac{1}{6} \\ \frac{13}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{pmatrix}$$

(b) $\det \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{pmatrix} \overset{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_2}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow$ A is invertible

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 5 & 4 \\ 6 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 3 & 5 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 25-24 & -(5-6) & 4-5 \\ -(15-12) & 5-3 & -(4-3) \\ 18-15 & -(6-3) & 5-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix} \Rightarrow$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{pmatrix}$

Using Jordan-Gauss reduction:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{array}\right) \overset{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 0 & 1 & 0 \\ 0 & 2 & -14 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\overset{R_3 \to R_3 - 2R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right) \longrightarrow$$ The matrix is not invertible (for example because $Ax=0$ has nontrivial solution)

Note that the same row operations show that $\det A = 0$

i.e. again A is not invertible

Problem 3

(a) For $a_{11} a_{22} a_{33} a_{44} a_{55} a_{66}$ the corresponding permutation is $(1\,2\,3\,4\,5\,6) \Rightarrow$ no inversion $\Rightarrow$ the sign is $\boxed{+}$

(b) The term $a_{12} a_{23} a_{35} a_{46} a_{52} a_{64}$ does not appear in the expansion of the determinant, because the entries a_{12} and a_{52} from the same column appear in this term,

because no entry from the first column appears in this term.

(c) For $a_{13} a_{24} a_{36} a_{42} a_{55} a_{61}$ the corresponding permutation is $(3\,4\,6\,2\,5\,1) \Rightarrow$ the inversions are

$(3,2), (3,1), (4,2), (4,1), (6,2), (6,5), (6,1), (2,1), (5,1) \Rightarrow$

we have 9 inversions $\Rightarrow$ it is an odd permutation $\Rightarrow$ the sign is $\boxed{\text{"}-\text{"}}$

Another way is to count the parity of the number of transpositions needed for transforming our permutation to the permutation $(1\,2\,3\,4\,5\,6)$:

$(3\,4\,6\,2\,5\,1) \to (1\,4\,6\,2\,5\,3) \to (1\,2\,6\,4\,5\,3) \to$

$(1\,2\,3\,4\,5\,6) \Rightarrow$ we used 3 transpositions $\Rightarrow$ the permutation is odd $\Rightarrow$ the sign is $\boxed{\text{"}-\text{"}}$

(d) For $a_{16}\,a_{25}\,a_{34}\,a_{43}\,a_{52}\,a_{61}$ the corresponding permutation is $(6\,5\,4\,3\,2\,1)$. Since the order is reversed (w.r.t. the standard order), all pairs (j_k, j_l) with $k<l$ are inversions $\Rightarrow$ the number of inversion is $\underbrace{\binom{6}{2}}_{\text{binomial coefficien}} = \frac{6\cdot 5}{2} = 15$

i.e. odd $\Rightarrow$ the sign is $\boxed{\text{"}-\text{"}}$

Another way Using transpositions:

$(6\,5\,4\,3\,2\,1) \to (1\,2\,3\,4\,5\,6) \to$ we used 3 transpositions $\Rightarrow$ an odd permutation $\Rightarrow$ the sign is $\boxed{\text{"}-\text{"}}$.

Problem 4 (a) $\det(3A^2B) = 3^7(\det A)^2 \det B = \boxed{3^7\cdot 5^2\cdot 7} = \boxed{382{,}725}$

here we actually use problem 6a)

(b) $\det(4A^{-1}B^2) = 4^7\cdot\frac{1}{\det A}(\det B)^2 = \boxed{\frac{4^7\cdot 7^2}{5}} = \boxed{\frac{802{,}816}{5}}$

(c) $\det(AB^{-2}) = \frac{\det A}{(\det B)^2} = \boxed{\frac{5}{49}}$

Problem 5

$$\begin{cases} 4x_1 - x_2 - x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 - 2x_3 = -1 \end{cases} \Rightarrow A = \begin{pmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & 2 \end{vmatrix} = 4 \begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 5 & -2 \end{vmatrix} = 4(4+6) +$$

$$+(4-15) - (-4-10) = 40 - 11 + 14 = 43$$

By Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & 2 \end{vmatrix}}{43} = \frac{(4+6) + (20+3) - (-20+2)}{43} = \frac{10+23+18}{43} =$$

here the first column is replaced by b

$$= \frac{51}{43}$$

$$x_2 = \frac{\begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & 2 \end{vmatrix}}{43} = \frac{4(20+3) - (4-15) - (-2-50)}{43} =$$

$$= \frac{92 + 11 + 52}{43} = \frac{155}{43}$$

$$x_3 = \frac{\begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix}}{43} = \frac{4(-2+20) + (-2-50) + (-4-10)}{43} =$$

$$= \frac{72 - 52 - 14}{43} = \frac{6}{43} \Rightarrow \boxed{\left(\frac{51}{43}, \frac{155}{43}, \frac{6}{43}\right)}$$

Problem 6

(a) $\det(\alpha A) = \sum_{\substack{\text{all permutations} \\ (j_1 \dots j_n)}} \epsilon(j_1 \dots j_n)(\alpha a_{1j_1})(\alpha a_{2j_2}) \dots (\alpha a_{nj_n}) =$

$= \alpha^n \sum_{\substack{\text{all permutations} \\ (j_1 \dots j_n)}} \epsilon(j_1 \dots j_n) a_{1j_1} \dots a_{nj_n} = \alpha^n \det A$

all terms have common factor α^n

Another way: We proved in class that for any matrix B and a scalar β

$$\det \begin{pmatrix} \vec{b_1} \\ \vdots \\ \beta \vec{b_i} \\ \vdots \\ \vec{b_n} \end{pmatrix} = \beta \det B \quad (*)$$

$$\alpha A = \begin{pmatrix} \alpha \vec{a_1} \\ \alpha \vec{a_2} \\ \vdots \\ \alpha \vec{a_n} \end{pmatrix} \underset{\substack{\text{using the} \\ \text{previous rule } n \text{ times}}}{\Rightarrow} \det(\alpha A) = \alpha \det \begin{pmatrix} \vec{a_1} \\ \alpha \vec{a_2} \\ \vdots \\ \alpha \vec{a_n} \end{pmatrix} =$$

$$= \alpha^2 \det \begin{pmatrix} \vec{a_1} \\ \vec{a_2} \\ \alpha \vec{a_3} \\ \vdots \\ \alpha \vec{a_n} \end{pmatrix} = \dots = \alpha^n \det A$$

(b) If A is 3×3 matrix and $A^T = -A$ then

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \Rightarrow$$

$$\det A = -a \begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b \begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} = -a(bc) +$$

$$+ b(ac) = -abc + abc = \boxed{0}$$

Another way is to prove more general item (c)

(c) Let us prove that $\det A = 0$

Indeed, we know that $\det A^T = \det A$

On the other hand, $A^T = -A \Rightarrow$ by item a)

$$\det A^T = (-1)^n \det A \Rightarrow$$

$$\det A = (-1)^n \det A$$

If n is odd, we get: $\det A = -\det A \Rightarrow$

$$\det A = 0$$

Problem 7

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \overset{R_1 \to R_1+R_2+R_3+R_4}{=} \begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} =$$

$$= (a+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \overset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}}{=} (a+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & a-1 \end{vmatrix} =$$

$$\underset{\substack{\text{we get} \\ \text{an upper diagonal} \\ \text{matrix}}}{=} (a+3)(a-1)^3 \quad \text{q.e.d.}$$

Problem 8

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \overset{\substack{C_2 \to C_2 - x_1 C_1 \\ C_3 \to C_3 - x_1 C_2}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) \\ 1 & x_3 - x_1 & x_3(x_3 - x_1) \end{vmatrix} =$$

$$\underset{\substack{\text{expansion on} \\ \text{along the first row}}}{=} \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} =$$

$= (x_2-x_1)(x_3-x_1)(x_2-x_3)$

Rem Note that in the proof above are intentially used an induction procedure, namely I proved that

$$\underbrace{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}_{\substack{3\times 3 \text{ Vandermonde} \\ \text{depending on } x_1, x_2, x_3}} = (x_2-x_1)(x_3-x_1) \underbrace{\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}}_{\substack{2\times 2 \text{ Vandermonde,} \\ \text{depending on } x_2 \text{ and } x_3}}$$

In general, using the same idea, you can prove by induction that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & x_2^{n-1} \\ 1 & \vdots & \vdots & & \vdots \\ 1 & & & & \\ 1 & x_n & x_n^2 & & x_n^{n-1} \end{vmatrix} = \underbrace{\prod_{1\le i<j\le n} (x_j - x_i)}_{\substack{\text{all possible} \\ \text{products of terms } x_j - x_i \text{ with } i<j}}$$

(b) V is nonsingular $\iff$ x_1, x_2, x_3 are <u>pair wise distinct</u>, i.e $x_i \neq x_j$ for $1 \le i < j \le 3$.