Homework #4 MATH309 - Solutions

Sec. 3.1 1

$x_1 = (8,6)^T$, $x_2 = (4,-1)^T$ in $\mathbb{R}^2$

1(a) $|x_1| = \sqrt{8^2+6^2} = \sqrt{100} = \boxed{10}$

$|x_2| = \sqrt{4^2+(-1)^2} = \boxed{\sqrt{17}}$

(b) $x_3 = (12,5)^T$

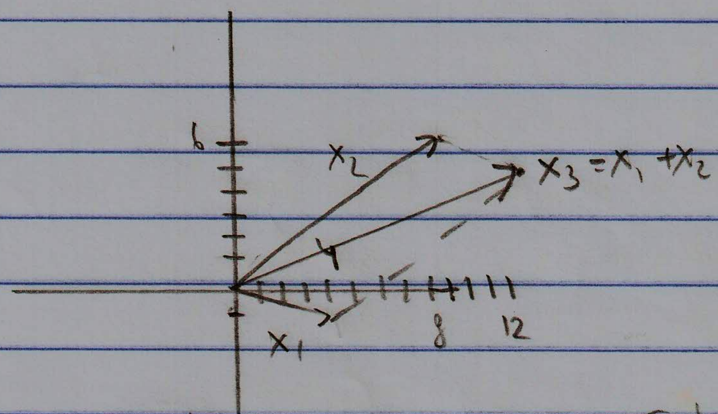
$|x_3| = \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = \boxed{13}$

$|x_1|+|x_2| = 10+\underbrace{\sqrt{17}}_{>4} > 14 > 13 = |x_3| \Rightarrow |x_1|+|x_2| > |x_3|$

(In general, there is the triangle inequality

$|x_1+x_2| \le |x_1|+|x_2|$)

(c)



Sec 3.1 3

Problem 3 Check all axioms (A1)-(A8)

(A1) $x = a+bi$, $y = c+id$

$x+y = (a+c)+(b+d)i$

$y+x = (c+a)+(d+b)i$

Since $a+c = c+a$ and $b+d = d+b$ then $x+y = y+x \Rightarrow$ (A1) holds

(here we used commutativity of reals under addition)

(A2) $x = a + bi,\ y = c + di,\ z = e + if$

$(x+y) + z = (a+c) + e + ((b+d)+f)i =$

$= a + (c+e) + (b + (d+f))i = x + (y+z) \Rightarrow$ (A2) holds

(Here we used associativity of reals under addition)

(A3) Let $0 = 0 + 0i \Rightarrow$ for any $x = a + ib$

$x + 0 = 0 + x = a + bi = x \Rightarrow$ (A3) holds

(A4) For $x = a + bi$ take $y = (-a) + i(-b) \Rightarrow$

$x + y = y + x = 0 \Rightarrow y$ can be taken as $-x$

(A5) For $x = a + bi$ and $y = c + di$

$\alpha(x+y) = \alpha((a+c) + (b+d)i) = \alpha(a+c) + \alpha(b+d)i$

$= \alpha a + \alpha c + (\alpha b + \alpha d)i = (\alpha a + \alpha bi) + (\alpha c + \alpha di) =$

$= \alpha(a + bi) + \alpha(c + di) = \alpha x + \alpha y$

(here we used the distributive law for reals)

(A6) Let $x = a + bi \Rightarrow (\alpha+\beta)x = (\alpha+\beta)(a+bi) =$

$= (\alpha+\beta)a + (\alpha+\beta)bi = (\alpha a + \beta a) + (\alpha b + \beta b)i =$

$= (\alpha a + \alpha bi) + (\beta a + \beta bi) = \alpha(a+bi) + \beta(a+bi) = \alpha x + \beta x$

(here we used the distributive law for reals)

(A7) Let $x = a+bi$

$$(\alpha\beta)x = (\alpha\beta)(a+bi) = (\alpha\beta)a + (\alpha\beta)bi =$$

$$= \alpha(\beta a) + \alpha(\beta b)i = \alpha(\beta a+\beta bi) = \alpha(\beta(a+bi)) = \alpha(\beta x)$$

(Here we used associativity of the multiplication of reals)

(A8) $1\cdot x = x$ for all $x \in V$

$$1\cdot(a+bi) = 1\cdot a + 1\cdot bi = a+bi = x$$

Sec 3.1 7. Assume that we have two elements 0_I and 0_{II} such that

$$x + 0_I = 0_I + x = x \quad \text{for any } x \qquad (1)$$

$$x + 0_{II} = 0_{II} + x = x \quad \text{for any } x \qquad (2)$$

In (1) take $x = 0_{II} \Rightarrow$

$$0_{II} + 0_I = 0_I + 0_{II} = 0_{II}$$

On the other hand from (2)

$$0_{II} + 0_I = 0_I \Rightarrow 0_I = 0_{II}, \text{ i.e}$$

zero element is unique.

Sec 3.1 10. Axiom A3) (an existence of zero element) does not hold. Indeed assume that 0 is a zero element (note that a priori we do not know whether $0 = (0,0)$)

Assume that $0 = (y_1, y_2)$

Then for any $x = (x_1, x_2)$

$(x_1, x_2) \oplus 0 = (x_1, x_2)$ (*) or equivalently

$x_1 + y_1 = x_1 \Rightarrow y_1 = 0$

$0 = x_2 \Rightarrow x_2$ must by equal to 0, but

we want that (*) will hold for arbitrary pair

of real numbers (x_1, x_2).

Sec 3.1 11. Clearly all axioms (A1)-(A2) related with addition hold (because the addition here is regular)

Look for the axioms involving multiplication by scalar

(A5) $\alpha \circ (x+y) \overset{?}{=} \alpha \circ x + \alpha \circ y$

$\alpha \circ ((x_1, x_2) + (y_1, y_2)) = \alpha \circ (x_1 + y_1, x_2 + y_2) = (\alpha(x_1 + y_1), x_2 + y_2)$

$\alpha \circ x + \alpha \circ y = (\alpha x_1, x_2) + (\alpha y_1, y_2) = (\alpha x_1 + \alpha y_1, x_2 + y_2) =$

$= (\alpha(x_1 + y_1), x_2 + y_2) \Rightarrow$ (A5) holds

(A6) $(\alpha + \beta) \circ x \overset{?}{=} \alpha \circ x + \beta \circ x$

Let $x = (x_1, x_2)$: $(\alpha + \beta) \circ (x_1, x_2) = ((\alpha + \beta) x_1, x_2)$

$\alpha \circ x + \beta \circ x = (\alpha x_1, x_2) + (\beta x_1, x_2) = (\alpha x_1 + \beta x_1, x_2 + x_2) =$

$= ((\alpha + \beta) x_1, 2x_2) \neq ((\alpha + \beta) x_1, x_2)$ if $x_2 \neq 0 \Rightarrow$

the axiom (A6) does not hold.

Section 3.2 In problems 1&2 I denote the set under examination by S

1(b) This is not a subspace, because it is not closed under addition

Take $x=(1,0)$, $y \in (0,1)$

$x \in S$, $y \in S$ (because $1 \cdot 0 = 0$, $0 \cdot 1 = 0$) But

$x+y=(1,1) \notin S$, because $1 \cdot 1 \neq 0$.

Rem S is a union of 2 lines (the coordinate axis). It is another explanation why it is not a subspace

1(a) This is a subspace of R^2

Indeed 1) S is closed under multiplication

If $(x_1,x_2) \in S \Rightarrow x_1 = 3x_2 \Rightarrow$ for any scaler α we have

$\alpha x_1 = 3\alpha x_2 \Rightarrow (\alpha x_1, \alpha x_2) \in S$

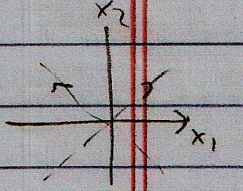
2) S is closed under addition

If $(x_1,x_2) \in S$ and $(y_1,y_2) \in S$ then $x_1 = 3x_2$, $y_1 = 3y_2 \Rightarrow$

$x_1+y_1 = 3(x_2+y_2) \Rightarrow (x_1+y_1, x_2+y_2) \in S$

Rem Note that in this case S is a line through the origin $\Rightarrow$ it is a subspace

1(e) $x_1^2 = x_2^2 \iff x_1 = x_2$ or $x_1 = -x_2$ (2 lines through the origin. This set is not closed under addition: for example



$x = (-1, 1) \in S$; $y = (1, 1) \in S$ but

$x + y = (0, 1) \notin S \Rightarrow$ it is not a subspace

Section 3.2

2(a) This is not a subspace

Explanation 1 S does not contain 0 (we proved that a subspace of a vectorspace must contain 0)

Explanation 2 S is not closed under a scalar multiplication: for example, $(1,0,0)^T \in S$ but $(2,0,0)^T \notin S$

$(2,0,0)^T = 2(1,0,0)^T$

Explanation 3 S is not closed under addition (the same counterexample as in explanation 2)

$(2,0,0)^T = (1,0,0)^T + (1,0,0)^T$

2(b) This is a subspace (it is a line through the origin)

1) S is closed under scalar multiplication:

If $(x_1, x_2, x_3) \in S \iff x_1 = x_2 = x_3 \Rightarrow$ for any scalar α

$\alpha x_1 = \alpha x_2 = \alpha x_3 \Rightarrow (\alpha x_1, \alpha x_2, \alpha x_3) \in S$

2) S is closed under addition

If $(x_1, x_2, x_3) \in S$ and $(y_1, y_2, y_3) \in S \Rightarrow$

$x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3 \Rightarrow$

$x_1 + y_1 = x_2 + y_2 = x_3 + y_3, \Rightarrow (x_1+y_1, x_2+y_2, x_3+y_3) \in S$

2 d) S is <u>not</u> a subspace (It is a union of two planes). It is not closed under addition

For example $x = (1, 0, 1) \in S$ and

$y = (0, 1, 1) \in S$ but

$x + y = (1, 1, 2)$ is not in S $(1 \neq 2)$

4 (c) To find the null space is the same as to solve the homogeneous equation

$$\begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Reduce to the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_2 \to -R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_3$ is free variable

$x_3 = \alpha$

$x_2 - x_3 = 0 \Rightarrow x_2 = \alpha$

$x_1 + 3x_2 - 4x_3 = 0 \Rightarrow x_1 + 3\alpha - 4\alpha = 0 \Rightarrow x_1 = \alpha \Rightarrow$

$$N(A) = \boxed{\{(\alpha,\alpha,\alpha)^T \in \mathbb{R}^3 \mid \alpha \in \mathbb{R}\} = \mathrm{Span}\left((1,1,1)^T\right)}$$

4 (d)
$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right) \overset{R_2 \to -R_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right) \overset{R_3 \to R_3 + R_2}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 \text{ and } x_4 \text{ are free variables}$$

Let $\boxed{x_2 = \alpha,\ x_4 = \beta}$ $\quad x_3 + 3\beta = 0 \Rightarrow \boxed{x_3 = -3\beta}$

$x_1 + x_2 - x_3 + 2x_4 = 0 \Rightarrow x_1 + \alpha + 3\beta + 2\beta = 0 \Rightarrow \boxed{x_1 = -5\beta - \alpha}$

$$\boxed{\begin{aligned} N(A) &= \{(-5\beta-\alpha, \alpha, -3\beta, \beta)^T \mid \alpha, \beta \in \mathbb{R}\} = \\ &= \{\alpha(-1,1,0,0)^T + \beta(-5,0,-3,1)^T \mid \alpha,\beta \in \mathbb{R}\} = \mathrm{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}\right) \end{aligned}}$$

5 (a) It is not a subspace of P_4, because it is not closed under addition: for example $p_1(x) = x^2$ and $p_2(x) = -x^2 + x$ are of even degree but $(p_1 + p_2)(x) = x^2 - x^2 + x = x$ is of odd degree.

5. (c) Yes, It is a subspace

If $p \in S \Rightarrow p_1(0) = 0 \Rightarrow \quad \alpha p_1(0) = 0 \Rightarrow \alpha p \in S$

If $p_1 \in S$ and $p_2 \in S$, then $p_1(0) = 0$, $p_2(0) = 0 \Rightarrow$

$p_1(0) + p_2(0) = 0 \Rightarrow (p_1 + p_2)(0) = 0 \Rightarrow p_1 + p_2 \in S$

11 (c) <u>Way one</u> : reduce the matrix

$\begin{pmatrix} -2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$ to a row echelon form

$$\begin{pmatrix} -2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & 3 & 4 \\ -2 & 1 & 2 \end{pmatrix} \overset{R_2 \to R_2 + 2R_1}{\sim} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 7 & 10 \end{pmatrix}$$

$\Rightarrow$ # of leading variables $= 2 =$ # of rows $\Rightarrow$ It is a spanning set

<u>Way two</u> check 2x2 minor of the same matrix

$$\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} = -6 - 1 = -7 \neq 0 \Rightarrow \boxed{\text{It is a spanning set}}$$

(d) <u>Way one</u> : reduce the matrix

$\begin{pmatrix} -1 & 1 & 2 \\ 2 & -2 & -4 \end{pmatrix}$ to row echelon form

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & -2 & -4 \end{pmatrix} \overset{R_1 \to -R_1}{\sim} \begin{pmatrix} 1 & -1 & -2 \\ 2 & -2 & -4 \end{pmatrix} \overset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

# of leading variables $= 1 <$ # of rows —

It is ~~not~~ a spanning set

__Way two:__ Check the 2×2 minors of the same matrix: $\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} = 0$ $\begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = 0$ (actually the third relation follows from the first two) $\Rightarrow$ It is not a spanning set

__Way three:__ the two last vectors are scalar multiple of the first one $\Rightarrow$ $\mathrm{Span}(v_1, v_2, v_3) = \underbrace{\mathrm{Span}\, v_1}_{\text{a line}} \neq \mathbb{R}^3$

12 (c) Check $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix}$

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} \overset{R_2 \to R_2 - R_1}{=} \begin{vmatrix} 2 & 3 & 2 \\ -1 & -1 & 0 \\ -2 & -2 & 0 \end{vmatrix} = 2 \begin{vmatrix} -1 & -1 \\ -2 & -2 \end{vmatrix} = 0 \Rightarrow$$

It is __not__ a spanning set in $\mathbb{R}^3$

(d) $\begin{vmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{vmatrix} = 0$ (the second column is -1 times the first one)

It is __not__ a spanning set in $\mathbb{R}^3$

e) It is not a spanning set in $\mathbb{R}^3$ because the number of vectors < 3.

14 a) Yes

$$\mathrm{Span}(x_1, \ldots, x_k) \overset{\text{is contained in}}{\subset} \mathrm{Span}(x_1, \ldots, x_{k+1})$$

(The span of the smaller set is contained in the span of the bigger set). Since $\{x_1, \ldots, x_k\}$ is a spanning set $\Rightarrow$ $\mathrm{Span}(x_1, \ldots, x_k) = V \Rightarrow \mathrm{Span}(x_1, \ldots, x_{k+1}) = V$

i.e $\{x_1, \ldots, x_{k+1}\}$ is also a spanning set of V

b) No $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a spanning set of $\mathbb{R}^2$

but if we remove one of the vectors it becomes nonspanning

(it spans a line, not the whole $\mathbb{R}^2$)