Homework assignment #5 solutions, MATH309

Problem 1

(a) $\begin{vmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ 1 & 2 & 2 \end{vmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 3R_3} \begin{vmatrix} 0 & -5 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -5 & -2 \\ -1 & 1 \end{vmatrix} = -5-2 = -7 \neq 0$

$\Downarrow$

$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ is a basis in R^3

(b) Since # vector > dim (i.e $4 > 3$) these vectors are $\boxed{\text{linearly dependent}}$

To choose a basis of their span transform to row echelon form:

$\begin{pmatrix} 3 & 1 & 6 & 1 \\ 2 & 3 & 11 & 10 \\ 1 & 2 & 7 & 3 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 2 & 3 & 11 & 10 \\ 3 & 1 & 6 & 1 \end{pmatrix} \underset{R_3 \to R_3 - 3R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & -5 & -15 & -8 \end{pmatrix} \sim$

$\overset{R_2 \to -R_2}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 5 & 15 & 8 \end{pmatrix} \overset{R_3 \to R_3 - 5R_1}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 28 \end{pmatrix}$

# of leading elements $= 3 \Rightarrow$ our set of vector is a spanning set in $\mathbb{R}^3$ and also since the leading elements are in the 1st, 2nd and 4th columns the 1st, 2nd, and 4th vectors form a basis, i.e.

$\boxed{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} \text{ form a basis of } R^3}$

$\boxed{\text{dim of span} = 3}$

1c) $\begin{vmatrix} 1 & 4 & 6 & -7 \\ 3 & 0 & 6 & 3 \\ 0 & 2 & 2 & -4 \\ 5 & 1 & 11 & 3 \end{vmatrix} \xrightarrow[R_4 \to R_4 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{vmatrix} 1 & 4 & 6 & -7 \\ 0 & -12 & -12 & 24 \\ 0 & 2 & 2 & -4 \\ 0 & -19 & -19 & 38 \end{vmatrix} =$

$= -12 \cdot 2 \cdot (-19) \begin{vmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow$ because the matrix contains equal rows

the vector are <u>linearly dependent</u>

To sort out the basis continue the same calculation to transform the corresponding matrix to the row echelon form (in the calculation of the determinant we used the same row operation as we used for such transformation to row echelon form)

$\begin{pmatrix} 1 & 4 & 6 & -7 \\ 3 & 0 & 6 & 3 \\ 0 & 2 & 2 & -4 \\ 5 & 1 & 11 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \underset{R_4 \to R_4 - R_2}{\overset{R_3 \to R_3 - R_2}{\sim}}$

$\sim \begin{pmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The leading elements correspond to the first and the second vectors $\Rightarrow$ the first and the second vectors form a basis of the span of our four vectors.

i.e $\boxed{\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}}$ form a basis

of the span and dim of span is $\boxed{2}$

d) The problem is equivalent to the same question for the vectors

$$\begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 11 \\ -12 \\ 8 \\ -7 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 6 \\ 4 \end{pmatrix} \text{ in } \mathbb{R}^4$$

Transform the matrix with these columns to the row echelon form

$$\begin{pmatrix} 3 & -1 & 11 & 2 \\ -2 & 3 & -12 & 1 \\ 4 & 2 & 8 & 6 \\ 1 & 5 & -7 & 4 \end{pmatrix} \overset{R_1 \leftrightarrow R_4}{\sim} \begin{pmatrix} 1 & 5 & -7 & 4 \\ -2 & 3 & -12 & 1 \\ 4 & 2 & 8 & 6 \\ 3 & -1 & 11 & 2 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 + 2R_1 \\ \sim \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix} \begin{pmatrix} 1 & 5 & -7 & 4 \\ 0 & 13 & -26 & 9 \\ 0 & -18 & 36 & -10 \\ 0 & -16 & 32 & -10 \end{pmatrix} \begin{matrix} R_2 \to \frac{1}{13} R_2 \\ \sim \\ R_3 \to -\frac{1}{18} R_3 \\ R_4 \to -\frac{1}{16} R_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 5 & -7 & 4 \\ 0 & 1 & -2 & 9/13 \\ 0 & 1 & -2 & \frac{5}{9} \\ 0 & 1 & -2 & \frac{5}{8} \end{pmatrix} \underset{R_4 \to R_4 - R_2}{\overset{R_3 \to R_3 - R_2}{\sim}} \begin{pmatrix} 1 & 5 & -7 & 4 \\ 0 & 1 & -2 & 9/13 \\ 0 & 0 & 0 & \frac{5}{9} - \frac{9}{13} \\ 0 & 0 & 0 & \underbrace{\frac{5}{8} - \frac{9}{13}}_{\text{can be killed in the next step}} \end{pmatrix} \begin{matrix} \\ \\ = -\frac{16}{117} \neq 0 \\ = -\frac{7}{117} \neq 0 \end{matrix}$$

$\Rightarrow$ there are free variables $\Leftrightarrow$ our set of matrices is $\boxed{\text{linearly dependent}}$

(another explanation: our calculations det $= 0$, because the row echelon form is upper triangular with a zero on the diagonal)

Leading element correspond to the 1st, 2nd and 4th rows $\Rightarrow$ the basis of the span is

$$\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \text{ and}$$

dim of the span $= \boxed{3}$

e) Write coordinates of the polynomial in the standard basis. Then the problem is reduced to the same problem for the following vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

# of vector > $\dim P_3 = 3 \Rightarrow$ they are linearly dependent

Consider the corresponding 3×4 matrix

$$\begin{pmatrix} -4 & -5 & -1 & 2 \\ 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \qquad (*)$$

this 3×3 minor is not equal to 0

$$\begin{vmatrix} -5 & -1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -8 \neq 0$$

$\Downarrow$

the last 3 polynomials form a basis in P_3

$2x^2-5,\ 2x-1,\ 2$ is a basis in P_3

dim of the span $= 3$

Rem You also can work as before transforming the 3×4 matrix $(*)$ to a row echelon form. Actually you can show that also (p_1, p_3, p_4) & (p_1, p_2, p_4) form a basis of

(f) Check the Wronskian at $x=0$:

$$Wr(e^{r_1x}, e^{r_2x}, e^{r_3x}) = \begin{vmatrix} e^{r_1x} & e^{r_2x} & e^{r_3x} \\ r_1e^{r_1x} & r_2e^{r_2x} & r_3e^{r_3x} \\ r_1^2e^{r_1x} & r_2^2e^{r_2x} & r_3^2e^{r_3x} \end{vmatrix}_x$$

At $x=0$ it is

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} \underset{R_3 \to R_3 - r_1 R_2}{\overset{R_2 \to R_2 - r_1 R_1}{\sim}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & r_2-r_1 & r_3-r_1 \\ 0 & r_2^2-r_2r_1 & r_3^2-r_1r_3 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & r_2-r_1 & r_3-r_1 \\ 0 & r_2(r_2-r_1) & r_3(r_3-r_1) \end{vmatrix} = \begin{vmatrix} r_2-r_1 & r_3-r_1 \\ r_2(r_2-r_1) & r_3(r_3-r_1) \end{vmatrix} =$$

common factor r_2-r_1 common factor r_3-r_1

$$= (r_2-r_1)(r_3-r_1) \begin{vmatrix} 1 & 1 \\ r_2 & r_3 \end{vmatrix} = (r_2-r_1)(r_3-r_1)(r_3-r_2) \neq 0$$

If r_1, r_2 & r_3 are pairwise distinct

$\Rightarrow e^{r_1t}, e^{r_2t}, e^{r_3t}$ are $\boxed{\text{linearly independent}} \Rightarrow$ $\boxed{\text{dim of span} = 3}$

(g) Note that $\cos 2x = 2\cos^2 x - 1$ (from trigonometry) $\Rightarrow$

$1 + \cos 2x - 2\cos^2 x = 0 \Rightarrow$

$1, \cos 2x, \cos^2 x$ are linearly dependent

On the other hand any 2 out of these 3 functions is linearly independent

Indeed, $Wr(1, \cos 2x) = \begin{vmatrix} 1 & \cos 2x \\ 0 & -2\sin 2x \end{vmatrix} =$

$= -2\sin 2x \neq 0$ for $x \neq \frac{\pi}{2} n$

$Wr(1, \cos^2 x) = \begin{vmatrix} 1 & \cos^2 x \\ 0 & \underbrace{-2\sin x \cos x}_{-\sin 2x} \end{vmatrix} = -\sin 2x \Rightarrow$ the same conclusion

$Wr(\cos 2x, \cos^2 x) = \begin{vmatrix} \cos 2x & \cos^2 x \\ -2\sin 2x & -\sin 2x \end{vmatrix} =$

$= -2\sin 2x \, (\underbrace{\cos 2x - 2\cos^2 x}_{1}) = -2\sin 2x$ - the same conclusion

Exercize 6 p. 137

Answer is yes

Proof $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \Rightarrow$

$c_1(x_1 + x_2) + c_2(x_2 + x_3) + c_3(x_1 + x_3) = 0 \Rightarrow$

$(c_1 + c_3)x_1 + (c_1 + c_2)x_2 + (c_2 + c_3)x_3 = 0 \Rightarrow$

since x_1, x_2, x_3 are linearly independent

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} (*) \overset{Eq2 - Eq1}{\Rightarrow} \begin{matrix} c_2 - c_3 = 0 \\ c_2 + c_3 = 0 \end{matrix} \Rightarrow c_3 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow$$

y_1, y_2, y_3 are linearly independent

Equivalently the matrix of the system $(*)$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \overset{R_2 - R_1}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow$$

$c_1 = c_2 = c_3 = 0 \Rightarrow$ the same conclusion

<u>Rem</u> In general if $x_1, x_2, \dots, x_n$ are linearly independent and

$$y_1 = s_{11} x_1 + s_{21} x_2 + \dots + s_{n1} x_n$$
$$\vdots$$
$$y_n = s_{1n} x_1 + s_{2n} x_2 + \dots + s_{n2} x_n$$

or in the matrix form

$$(y_1 \dots y_n) = (x_1 \dots x_n) S \quad , \quad S = (s_{ij})$$

then $y_1, \dots, y_n$ are linearly independent $\Leftrightarrow \det S \neq 0$

Exercize 12, page 138

(a) If there exist c_1 and c_2 such that $c_1 2x + c_2 |x| = 0$ for any $x \in [-1, 1]$

then, first, subsdidute $x = 1$:

$$2c_1 + c_2 = 0$$

second, subsdidute $x = -1$

$$-2c_1 + c_2 = 0$$

$$\begin{cases} 2c_1 + c_2 = 0 \\ -2c_1 + c_2 = 0 \end{cases} \overset{Eq1+Eq2}{\Rightarrow} c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow 2x \text{ and } |x|$$

are linearly independent on $C[-1, 1]$

(b) For $x \geq 0$ $|x| = x \Rightarrow$ on $[0, 1]$

$$2x - 2|x| = 0$$ (i.e we can take

$c_1 = 1$, $c_2 = -2$) $\Rightarrow 2x$ and $|x|$ are linearly dependent on $C[0, 1]$

Exercize 13 If $v_1, \ldots, v_n$ is a set of vectors such that say $v_1 = 0$ then for any scalar $c_1 \neq 0$ $c_1 v_1 = c_1 \cdot 0 = 0 \Rightarrow v_1, \ldots, v_n$ are linearly dependent.

Problem 5 (from the assignment sheet)

(a)

$$\begin{pmatrix} a+3b+c \\ -b+2c \\ 3a+2b-2c \\ -b+c \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}}_{v_2} + c \underbrace{\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}}_{v_3}$$

Therefore

$$S = \text{Span}(v_1, v_2, v_3)$$

Check if v_1, v_2, v_3 are linearly independent:

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} \overset{R_3 \to R_3 - 3R_1}{\sim} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -7 & -5 \\ 0 & -1 & 1 \end{pmatrix} \overset{R_2 \to -R_2}{\sim} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & -7 & -5 \\ 0 & -1 & 1 \end{pmatrix} \sim$$

$$\underset{R_4 \to R_4 + R_1}{\overset{R_3 \to R_3 + 7R_1}{\sim}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -19 \\ 0 & 0 & -1 \end{pmatrix}$$

$\to$ no free variables (or the 3×3 minor consisting of first 3 rows is nonzero)

$\Rightarrow$ linearly independent $\Rightarrow$ v_1, v_2, v_3 is a basis

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} \text{ is a basis of } S$$

(b) Passing to coordinates w.r.t. the standard basis:

$$\begin{pmatrix} b+2a \\ 2b+a \\ a \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow$$ x^2+x+2, $2x+1$ is a spanning set of S

Vectors $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ are linearly independent

Because for example $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 \neq 0 \Rightarrow$ polynomials

x^2+x+2, $2x+1$ are linearly independent $\Rightarrow$

$\boxed{x^2+x+2,\ 2x+1 \text{ form a basis of } S}$

(c) $P_3 = \{ ax^2+bx+c \mid a, b, c \text{ in } \mathbb{R} \}$

$p(1) = 0 \Leftrightarrow a+b+c = 0 \Leftrightarrow a = -b-c \Leftrightarrow$

$p(x) = (-b-c)x^2 + bx + c = -b(x^2-x) - c(x^2-1) \Rightarrow$

x^2-x, x^2-1 is a spanning set of S

Besides x^2-x, x^2-1 are linearly independent because the corresponding coordinate vectors $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(w.r.t. to the standard basis of P_3) are linearly independent $\Rightarrow$ $\boxed{x^2-x \text{ and } x^2-1 \text{ is a basis in } P_3}$

(d) $p(0)=p(1)=0 \iff \begin{matrix} a+b+c=1 \\ c=0 \end{matrix} \iff \begin{matrix} a+b=1 \\ c=0 \end{matrix}$

$\Rightarrow c=0, \quad a=-b \Rightarrow$

$$p(x) = -bx^2+bx = -b(x^2-x)$$

$\Rightarrow \boxed{x^2-x \text{ is a basis of } S}$

Problem 6 $\quad x_1v_1+x_2v_2+x_3v_3 = y_1u_1+y_2u_2+y_3u_3 \iff$

$$\underbrace{(v_1\ v_2\ v_3)}_{V}\underbrace{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}_{x} = \underbrace{(u_1\ u_2\ u_3)}_{U}\underbrace{\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}}_{y} \Rightarrow$$

$$Vx = Uy \Rightarrow y = U^{-1}Vx \Rightarrow$$

The transition matrix is $U^{-1}V$ $\qquad U = \begin{pmatrix}0 & -2 & 1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}$

$U = \begin{pmatrix}0 & -2 & 1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{pmatrix} \qquad V = \begin{pmatrix}1 & -1 & 2\\ 2 & 2 & 4\\ 1 & 0 & 0\end{pmatrix}$

$$\det U = \begin{vmatrix}0 & -2 & 1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix}0 & -2 & 1\\ 2 & 3 & 0\\ 1 & 2 & 0\end{vmatrix} = \begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix} = 4-3 = 1 \neq 0 \Rightarrow$$

invertible

$$\operatorname{adj} U = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & -3 \\ -1 & -1 & 2 \\ -1 & -2 & 4 \end{pmatrix}$$

$$\Downarrow$$

$$U^{-1} = \frac{1}{\det U} \operatorname{adj} U = \begin{pmatrix} 1 & 2 & -3 \\ -1 & -1 & 2 \\ -1 & -2 & 4 \end{pmatrix}$$

2) transition matrix $= U^{-1}V = \begin{pmatrix} 1 & 2 & -3 \\ -1 & -1 & 2 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 4 & 0 & 0 \end{pmatrix} =$

$$= \begin{pmatrix} 1+4-12 & -1+4 & 2+8 \\ -1-2+8 & 1-2 & -2-4 \\ -1-4+16 & 1-4 & -2-8 \end{pmatrix} = \boxed{\begin{pmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{pmatrix}}$$

Another method is to transform the matrix $(U \mid V)$ to the reduced row echelon form

then $(U \mid V) \longrightarrow (I, U^{-1}V)$

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 & 2 & 4 \\ 1 & 0 & 1 & 4 & 0 & 0 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 2 & 1 & 1 & 2 & 2 & 4 \\ 0 & -2 & 1 & 1 & -1 & 2 \end{array}\right) (\leftarrow)$$

$$\overset{R_2 \to R_2 - 2R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & -6 & 2 & 4 \\ 0 & -2 & 1 & 1 & -1 & 2 \end{array}\right) \overset{R_3 \to R_3 + 2R_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & -6 & 2 & 4 \\ 0 & 0 & -1 & -11 & 3 & 10 \end{array}\right) \overset{R_3 -}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & -6 & 2 & 4 \\ 0 & 0 & 1 & 11 & -3 & -10 \end{array}\right) \sim$$

$$\underset{R_2 \to R_2 + R_3}{\overset{R_1 \to R_1 - R_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 3 & 10 \\ 0 & 1 & 0 & 5 & -1 & -6 \\ 0 & 0 & 1 & 11 & -3 & -10 \end{array}\right)$$

$$\underbrace{\hphantom{-7 \quad 3 \quad 10}}_{U^{-1}V}$$ (the same answer)

b) If $x = 3v_1 - 4v_2 + v_3 \Rightarrow$

the coordinates of x w.r.t. u_1, u_2, u_3 are

$$U^{-1}V \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -21 - 12 + 10 \\ 15 + 4 - 6 \\ 33 + 12 - 10 \end{pmatrix} = \begin{pmatrix} -23 \\ 13 \\ 35 \end{pmatrix}$$