Homework assignment #7 Solutions MATH 309

Problem 1

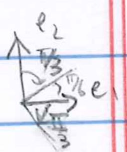
$$A = \begin{pmatrix} 2 & -3 & 1 \\ -4 & 0 & 1 \end{pmatrix}$$

Problem 2

In all items you have to find $L(e_1)$ and $L(e_2)$

(a) $L(e_1) = \begin{pmatrix} \cos(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \Rightarrow A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$L(e_2) = \begin{pmatrix} \cos\frac{\pi}{6} \\ \sin\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$

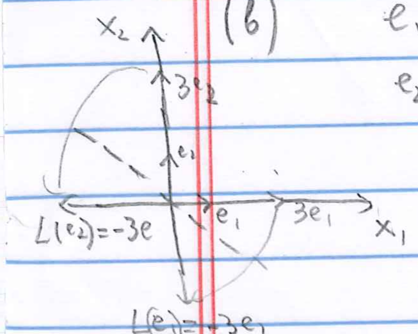


(b) $e_1 \xrightarrow{\text{tripling}} 3e_1 \xrightarrow{\text{reflecting}} -3e_2 \Rightarrow L(e_1) = -3e_2 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

$e_2 \xrightarrow{\text{tripling}} 3e_2 \xrightarrow{\text{reflecting}} -3e_1 \Rightarrow L(e_2) = -3e_1 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$$\Downarrow$$

$$A = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

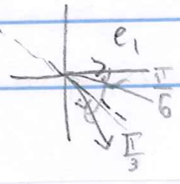


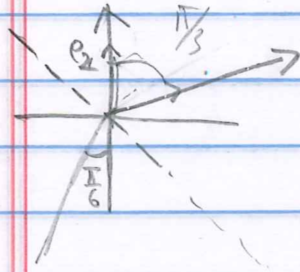
1c) The required matrix $=$ $\underbrace{\text{the product of the}}$ matrix of (b) on the matrix of (a) $=$

$$= \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix}$$

Another way

$$e_1 \xrightarrow{\text{rotating}} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \xrightarrow{\text{tripling}} \begin{pmatrix} \frac{3}{2} \\ -\frac{3\sqrt{3}}{2} \end{pmatrix} \xrightarrow{\text{reflecting}} \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ -\frac{3}{2} \end{pmatrix}$$





$$e_2 \underset{\text{rotating}}{\to} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \xrightarrow{\text{tripling}} \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} \to \begin{matrix} -\frac{3}{2} \\ -\frac{3\sqrt{3}}{2} \end{matrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix}$$

3. $$[L(x)]_{\{b_1, b_2, b_3\}} = \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + 4x_2 \\ -x_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -3 \\ 2 & 4 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to$$

4. Let $$\underbrace{U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}}_{\text{the transition matrix from } \{u_1, u_2, u_3\} \text{ to } \{e_1, e_2, e_3\}}, \quad \underbrace{A = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & -4 \end{pmatrix}}_{\text{the matrix representing } L \text{ in the standard bases}}, \quad \underbrace{B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}}_{\text{the transition matrix from } \{b_1, b_2\} \text{ to } \{e_1, e_2\}}$$

The corresponding diagram

$$\begin{tikzcd}
\text{standard basis of } \mathbb{R}^3 \arrow[r, "A"] & \text{standard basis of } \mathbb{R}^2 \\
\{u_1, u_2, u_3\} \arrow[u, "U"] \arrow[r, "C"'] & \{b_1, b_2\} \arrow[u, "B"']
\end{tikzcd}$$

$C \to$ the required matrix

$$\Downarrow$$

$$C = B^{-1} A U$$

$$AU = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1+2 & -1+2-2 & 1+1-2 \\ 1+4 & 1-4 & -1-4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 5 & -3 & -5 \end{pmatrix}; \quad B^{-1} = \frac{1}{-1+2} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$B^{-1}AU = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 5 & -3 & -5 \end{pmatrix} = \begin{pmatrix} -11 & 7 & 10 \\ 6 & -4 & -5 \end{pmatrix}$$

5. a) $L(1) = 2$

$L(x) = 0 + 2 = 2$

$L(x^2) = 1 + 2 = 3$

$L(x^3) = 3x + 2$

$\Downarrow$

The answer is $\begin{pmatrix} 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

b) The matrix representing L w.r.t. the basis $\{x^3, x^2, x, 1\}$ and $\{1, x\}$ is obtained from the matrix of item a by reversing the order of columns, because we reverse the order in the basis of the domain space, i.e we get

$$A = \begin{pmatrix} 2 & 3 & 2 & 2 \\ 3 & 0 & 0 & 0 \end{pmatrix} \qquad (*)$$

To obtain the matrix representation of L w.r.t $\{x^3, x^2, x, 1\}$ and $\{1+x, 1-x\}$ we have to multiply the matrix A in $(*)$ by the inverse of the transition matrix S from $\{1+x, 1-x\}$ to $\{1, x\}$ (see the diagram) which is $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\begin{tikzcd} \{x^3, x^2, x, 1\} \arrow[r, "A"] \arrow[dr, "C"'] & \{1, x\} \\ & \{1+x, 1-x\} \arrow[u, "S"'] \end{tikzcd}$$

the required matrix

$\Downarrow$

$$C = S^{-1}A = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 & 2 \\ 3 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 & 2 \\ 3 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 5 & 3 & 2 & 2 \\ -1 & 3 & 2 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 5/2 & 3/2 & 1 & 1 \\ -1/2 & 3/2 & 1 & 1 \end{pmatrix}}$$

6.

$$\begin{tikzcd}
\{e_1, e_2\} \arrow[r, "A"] & \{e_1, e_2\} \\
\{u_1, u_2\} \arrow[u, "S"] \arrow[r, "B"'] & \{u_1, u_2\} \arrow[u, "S"]
\end{tikzcd}$$

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$$

$$B = S^{-1}AS = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 3 & 1 \\ -7 & 7 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 0 & 14 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 7 \end{pmatrix}$$

7 In a similar way

$$A = \begin{pmatrix} 15 & -11 & 5 \\ 20 & -15 & 8 \\ 8 & -7 & 6 \end{pmatrix}$$

$$S = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

We have to calculate $S^{-1}AS$

Transform $(S \mid AS)$ to the reduced row echelon form.

$$AS = \begin{pmatrix} 15 & -11 & 5 \\ 20 & -15 & 8 \\ 8 & -7 & 6 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 30-33+5 & 45-44+5 & 15-22+15 \\ 40-45+8 & 60-60+8 & 20-30+24 \\ 16-21+6 & 24-28+6 & 8-14+18 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 8 \\ 3 & 8 & 14 \\ 1 & 2 & 12 \end{pmatrix}$$

$$\left(S \mid AS\right) = \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 2 & 6 & 8 \\ 3 & 4 & 2 & 3 & 8 & 14 \\ 1 & 1 & 3 & 1 & 2 & 12 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 2 & 12 \\ 3 & 4 & 2 & 3 & 8 & 14 \\ 2 & 3 & 1 & 2 & 6 & 8 \end{array}\right) \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 2 & 12 \\ 0 & 1 & -7 & 0 & 2 & -22 \\ 0 & 1 & -5 & 0 & 2 & -16 \end{array}\right)$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{cccccc} 1 & 1 & 3 & 1 & 2 & 12 \\ 0 & 1 & -7 & 0 & 2 & -22 \\ 0 & 0 & 2 & 0 & 0 & 6 \end{array}\right) \underset{R_3 \to \frac{1}{2}R_3}{\xrightarrow{R_1 \to R_1 - R_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & 1 & 0 & 34 \\ 0 & 1 & -7 & 0 & 2 & -22 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right)$$

$$\underset{R_2 \to R_2 + 7R_3}{\overset{R_1 \to R_1 - 10R_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right)$$

8.

$$\begin{tikzcd} \{u_1, u_2\} \arrow[r, "B"] & \{u_1, u_2\} \\ \{v_1, v_2\} \arrow[u, "S"] \arrow[r, "A"] & \{v_1, v_2\} \arrow[u, "S"'] \end{tikzcd}$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$A = S^{-1} B S$, where S is the transition matrix from (v_1, v_2) to (u_1, u_2) =>

If $U = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ then

$S = U^{-1} V = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$

$$U^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$U^{-1}V = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix}$$

$$S^{-1} = 2\,\frac{1}{-3-1}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$A = S^{-1}BS = \frac{1}{4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix} =$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1-3 & 1+9 \\ -1-1 & -1+3 \end{pmatrix} =$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 10 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$