Homework Assignment 8 in MATH309-Spring 2013, ©Igor Zelenko

due March 27, 2013 . Show your work in all exercises (you may earn 30 points of bonus)

Sections covered are 6.1, 6.2, and 6.3: Eigenvalues of matrices/linear operators, eigenvectors and eigenspaces, relations to the determinant and the trace;algebraic and geometric multiplicity of an eigenvalue, diagonalizable and defective matrices/operators; the exponential of the matrix and its relation to the solution of initial value problems for systems of ordinary differential equations with constant coefficients

(1) In items (a)-(e) below I give a matrix A and a vector $\mathbf{x}_0$ [1] together (maybe) with some additional information about the characteristic polynomial or some of eigenvalues of A . In each item answer the following questions:
   (i) Find all eigenvalues (that are not given);
   (ii) Find the eigenspaces corresponding to each eigenvalue;
   (iii) Determine weather the matrix A is diagonizable or defective (i.e. non-diagonizable);
   (iv) In the case when the matrix A is diagonizable find the diagonal matrix D which is similar to A and a matrix X which diagonalizes A (in other words, factor A into the product XDX^{-1}, where D is diagonal by writing explicitly what is D and X);
   (v) In all cases find A^3;
   (vi) In all cases find e^{tA}
   (vii) Find the solution of the initial value problem $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \mathbf{x}_0$.
   (a) $A = \begin{pmatrix} -7 & 6 \\ -15 & 12 \end{pmatrix}$, $x_0 = (1, -1)^T$;
   (b) $A = \begin{pmatrix} -13 & 40 & -10 \\ -9 & 27 & -7 \\ -17 & 50 & -14 \end{pmatrix}$, $\mathbf{x}_0 = (3, -4, -2)^T$, and it is known that -3 and 1 are eigenvalues of A (Hint: you can answer all questions for A without calculating its characteristic polynomial);
   (c) $A = \begin{pmatrix} 7 & -27 & -42 \\ -2 & 10 & 14 \\ 2 & -9 & -13 \end{pmatrix}$, $\mathbf{x}_0 = (1, -1, 1)^T$, and it is known that $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$ is a characteristic polynomial of A;
   (d) $A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, $\mathbf{x}_0 = (2, 1, 0, -2)^T$;
   (e) $A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 12 & 10 \\ 4 & -17 & -14 \end{pmatrix}$, $\mathbf{x}_0 = (-1, 2, 3)$ (for this matrix items (v)-(vii) are **bonus 15 points in total**).

(2) For the matrix A from the item (a) of the previous problem find a matrix B such that $B^2 = A$;

(3) Solve any 3 out of the following 4 problems from page 294 of the textbook: Exercise 3,4,8,9; if you solve all 4 of them you get **bonus 5 points**;

(4) Exercise 12, page 294;

(5) Let A be a 2×2 matrix . If $\text{tr}A = 1$ and $\det(A) = -20$, what are the eigenvalues of A? Is A diagonizable?

(6) Solve any 3 out of the following 4 problems: Exercise 18, 24, 26, 29 page 295; if you solve all 4 of them you get **bonus 5 points**;

(7) Let A be a 5×5 matrix.
   (a) If λ is an eigenvalue of (algebraic) multiplicity 4 and $A - \lambda I$ has rank 1, is A defective? Explain your answer;
   (b) If λ is an eigenvalue of (algebraic) multiplicity 3 and $A - \lambda I$ has rank 3, is A defective? Explain your answer;
   (c) **bonus 5 points** Is it possible that λ is an eigenvalue of (algebraic) multiplicity 3 such that $A - \lambda I$ has rank 1? Explain your answer.

[1] the vector $\mathbf{x}_0$ is used in item (vi) only