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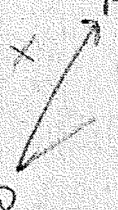
Homework #9 solution MATH 309

Problem 1 By Cauchy - Scwartz - Bunyakovski

$$|x^T y| \le \|x\| \, \|y\| = 3 \cdot 4 = 12 \Rightarrow$$

$$\boxed{|x^T y| \le 12} \Leftrightarrow -12 \le x^T y \le 12$$

Problem 2 The required point is defined by the vector projection p of the vector $x = (1,2,-3,4)^T$ onto the vector $y = (-2,3,-4,-2)$



(this point is the endpoint of this vector projection, if we take the origin as the starting point)

$$p = \frac{\langle x, y\rangle}{\|y\|^2} y = \frac{-2-6+12-8}{4+9+16+4}(-2,3,-4,-2)^T =$$

$$= \frac{-4}{33}(-2,3,-4,-2)^T = \boxed{\left(\frac{8}{33}, -\frac{12}{33}, \frac{16}{33}, \frac{8}{33}\right)^T}$$

The distance from A to $L = \|x-p\| = \left\|\left(1-\frac{8}{33}, -2+\frac{12}{33}, -3-\frac{16}{33}, 4-\frac{8}{33}\right)^T\right\| =$

$$= \left\|\left(\frac{25}{33}, -\frac{54}{33}, -\frac{115}{33}, \frac{124}{33}\right)\right\| = \frac{1}{33}\sqrt{(25)^2+(54)^2+(115)^2+(124)^2} =$$

$$= \boxed{\frac{1}{33}\sqrt{32142}}$$

Problem 3

$$\text{distance} = \frac{|2(3-2)+3(-1+1)+4(4-5)+5(-2+4)|}{\sqrt{2^2+3^2+4^2+5^2}} =$$

$$= \frac{|2-4+10|}{\sqrt{4+9+16+25}} = \boxed{\frac{8}{\sqrt{54}}}$$

Problem 4 Exercize 18 p. 213

(b) I will use the notation of the book here (in class we used $x \cdot y$ instead of $x^T y$ for the standard scalar product in $\mathbb{R}^n$)

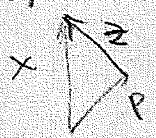
Prove that $p^T(x-p) = 0$ or equivalently

$$p^T x = p^T p$$

$$p^T x = \left(\underbrace{\frac{x^T y}{y^T y}}_{\text{constant}} y\right)^T x = \left(\frac{x^T y}{y^T y}\right) y^T x \underset{y^T x = x^T y}{=} \frac{(x^T y)^2}{y^T y} \Rightarrow \text{Indeed } p^T x = p^T p$$

$$p^T p = \left(\underbrace{\frac{x^T y}{y^T y}}_{\text{constant}} y\right)^T \left(\underbrace{\frac{x^T y}{y^T y}}_{\text{constant}}\right) y = \frac{(x^T y)^2}{(y^T y)^2} y^T y = \frac{(x^T y)^2}{y^T y}$$

(b) Since $p \perp z$ by Pythagora's Thm



$$\|x\|^2 = \|p\|^2 + \|z\|^2 = 6^2 + 8^2 = 36 + 64 = 100 \Rightarrow \|x\| = 10$$

Problem 5

(a) (i) $\langle y, z \rangle = 1 \cdot 1 + 3 \cdot (-3) + 1 \cdot 1 + 1 \cdot 1 = 1 - 9 + 1 + 1 = -6$

(ii) $\|y - z\| = \sqrt{(1-1)^2 + (3-(-3))^2 + (1-1)^2 + (1-1)^2} = 6$

(iii) $\cos\theta = \frac{\langle y, z \rangle}{\|y\| \cdot \|z\|} = \frac{-6}{\sqrt{1+9+1+1}\,\sqrt{1+9+1+1}} = -\frac{6}{12} = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

(IV) $\text{proj}_z y = \frac{\langle y, z \rangle}{\|z\|^2} z = \frac{-6}{12}(1, -3, 1, 1)^T = (-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})^T$

(b) (i) $\langle y, z \rangle = 1 \cdot 1 \cdot 3 + 3 \cdot (-3) \cdot 1 + 1 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 3 = 3 - 9 + 3 + 3 = 0$

(ii) $\|y - z\| = \sqrt{(1-1)^2 3 + (3-(-3))^2 \cdot 1 + (1-1)^2 \cdot 3 + (1-1)^2 \cdot 3} = 6$

(iii) $\cos\theta = \frac{\langle y, z \rangle}{\|y\|\|z\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$ i.e. $y \perp z$

(IV) $\text{proj}_z y = \frac{\langle y, z \rangle}{\|z\|^2} \cdot z = 0$

3). (c) The answers are exactly the same as in the item (a)

(one says that there is an isomorphism between inner product spaces $\mathbb{R}^4$ and $\mathbb{R}^{2\times 2}$ given by

$$(x_1,x_2,x_3,x_4)^T \mapsto \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix})$$

This isomorphism send the vectors y and z of item (a) to the corresponding matrices of the item (c)

(d) (i) $$\langle y,z\rangle = \int_0^1 e^{-2x}e^{2x}\,dx = \int_0^1 dx = 1$$

(ii) $$\|y-z\|^2 = \int_0^1 \left(e^{-2x}-e^{2x}\right)^2 dx = \int_0^1 \left(e^{4x}-2+e^{-4x}\right)dx =$$

$$= \frac{1}{4}\left(e^{4x}-e^{-4x}\right)\Big|_0^1 - 2 = \frac{e^4-e^{-4}}{4} - 2 = \frac{1}{2}\underbrace{\sinh}_{\text{hyperbolic sin}} 4 - 2 \Rightarrow$$

$$\boxed{\|y-z\| = \sqrt{\frac{e^4-e^{-4}}{4}-2} = \sqrt{\frac{1}{2}\sinh 4 - 2}}$$

(iii) $$\cos\theta = \frac{\langle y,z\rangle}{\|y\|\,\|z\|}$$

$$\|y\| = \sqrt{\int_0^1 e^{-4x}dx} = \sqrt{-\frac{1}{4}e^{-4x}\Big|_0^1} = \frac{\sqrt{1-e^{-4}}}{2}$$

$$\|z\| = \sqrt{\int_0^1 e^{4x}dx} = \sqrt{\frac{1}{4}e^{4x}\Big|_0^1} = \frac{\sqrt{e^4-1}}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{\frac{\sqrt{(1-e^{-4})(e^4-1)}}{4}} = \frac{4}{\sqrt{\underbrace{e^4-1-1+e^{-4}}_{(e^2-e^{-2})^2}}} = \frac{4}{e^2-e^{-2}} = \frac{2}{\sinh 2}$$

$$\Rightarrow \boxed{\theta = \arccos\frac{4}{e^2-e^{-2}} = \arccos\frac{2}{\sinh 2}}$$

(iv) $\text{proj}_z y = \frac{\langle y, z\rangle}{\|z\|^2} \cdot z = \frac{1}{\frac{e^4-1}{4}} \cdot e^{2x} = \frac{4}{e^4-1} e^{2x}$

1e) (i) $\langle y, z\rangle = \int_{-1}^{1} x^2 \cdot x^3 \, dx = \underbrace{\int_{-1}^{1} x^5 \, dx}_{\text{odd function}} = \boxed{0}$

(ii) $\|y-z\|^2 = \int_{-1}^{1} (x^2 - x^3)^2 \, dx = \int_{-1}^{1} (x^4 - 2x^5 + x^6) \, dx =$

$= \frac{x^5}{5} - 2\frac{x^6}{6} + \frac{x^7}{7} \Big|_{-1}^{1} = \frac{2}{5} + \frac{2}{7} = \frac{24}{35} \Rightarrow$

$\|y-z\| = \boxed{\sqrt{\frac{24}{35}} = \frac{2\sqrt{6}}{\sqrt{35}}}$

(iii) $\cos\theta = \frac{\langle y, z\rangle}{\|y\| \|z\|} \underset{\text{by (i)}}{=} 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$ $(y \perp z)$

(iv) $\text{proj}_z y = \boxed{0}$

1f) (i) $\langle y, z\rangle = (-1)^2 \cdot (-1)^3 + \left(-\frac{1}{2}\right)^2 \cdot \left(-\frac{1}{2}\right)^3 + 0 + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + 1^2 \cdot 1^3 = \boxed{0}$

cancelled

(ii) $\|y-z\|^2 = \underbrace{\left((-1)^2 - (-1)^3\right)^2}_{2} + \underbrace{\left(\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^3\right)^2}_{\frac{1}{4}+\frac{1}{8}} + (0-0)^2 + \underbrace{\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3\right)^2}_{\frac{1}{4}-\frac{1}{8}} + \underbrace{\left(1^2 - 1^3\right)^2}_{0} =$

$4 + \frac{9}{64} + \frac{1}{64} = 4 + \frac{10}{64} = 4 + \frac{5}{32} = \frac{133}{32} \Rightarrow \|y-z\| = \boxed{\sqrt{\frac{133}{32}} = \frac{1}{4}\sqrt{\frac{133}{2}}}$

(iii) $\cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

(iv) $\text{proj}_z y = \boxed{0}$

Ex. 18 p.240

Assume that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (*)

Note that $\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u,u \rangle + \langle v,u \rangle + \langle u,v \rangle + \langle v,v \rangle =$

$= \|u\|^2 + 2\langle u,v \rangle + \|v\|^2 \overset{(*)}{\Rightarrow}$

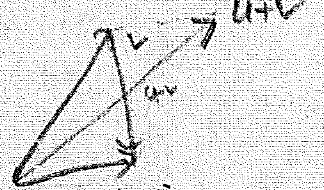
$\cancel{\|u\|^2} + 2\langle u,v \rangle + \cancel{\|v\|^2} = \cancel{\|u\|^2} + \cancel{\|v\|^2} \Rightarrow \langle u,v \rangle = 0 \Rightarrow u \perp v$ q.e.d

Ex. 26 p240

$\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v \rangle + \langle u-v, u-v \rangle = \langle u,u \rangle + \cancel{2\langle u,v \rangle} + \langle v,v \rangle + \langle u,u \rangle -$

$-\cancel{2\langle u,v \rangle} + \langle v,v \rangle = 2\|u\|^2 + 2\|v\|^2$, q.e.d.

Geometric interpretation: $\|u+v\|$ and $\|u-v\|$ are the lengths of the diagonals of the parallelogram generated by u and v $\Rightarrow$



the formula says that the sum of squares of all sizes of a parallelogram is equal to the sum of squares of the diagonal.