NAME (printed neatly)________________ QUIZ#11 GRADE ________

Directions for taking quizzes: the same as in the previous quizzes.

1. Decide which of the following sets are groups under the given operations, justify your answer. If your answer is positive clearly indicate what is the identity of the group and what is the inverse of each element :

   (a) The set of all non-zero real numbers of the form $a + b\sqrt{3}$ were a and b are rational numbers, under multiplication;

   (b) The set of all non-zero real numbers of the form $a + b\sqrt{3}$ were a and b are integer numbers, under multiplication;

   (c) The set of non-zero real numbers under the operation $*$ defined by $a * b = \frac{1}{5}ab$.

2. Let a, b be elements of a group G. Find (in terms of a and b) an expression for the solution of the equation $bxab^{-1} = b^{-1}ab$ (simplify it as much as possible).

1(a) The answer is $\boxed{\text{yes}}$: 1) the set is closed under the multiplication

If $x_1 = a_1 + b_1\sqrt{3}$, $x_2 = a_2 + b_2\sqrt{3}$ where $a_1, b_1, a_2, b_2 \in \mathbb{Q}$

then $x_1 x_2 = (a_1 + b_1\sqrt{3})(a_2 + b_2\sqrt{3}) = \underbrace{a_1 a_2 + 3 b_1 b_2}_{\in \mathbb{Q}} + \underbrace{(b_1 a_2 + a_1 a_2)}_{\in \mathbb{Q}}\sqrt{3}$

$\Rightarrow$ $x_1 x_2$ belongs to the same set.

Here we can either continue to check the axioms of the group or use Theorem 5.1.6 about the characterization of a subgroup

We choose the second way:

(1) 1 belongs to the set $1 = 1 + 0\sqrt{3}$ $\Rightarrow \boxed{e = 1}$

(2) If $x = a + b\sqrt{3}$ then $\frac{1}{x} = \frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{(a+b\sqrt{3})(a - b\sqrt{3})} = \frac{a - b\sqrt{3}}{a^2 - 3b^2}$

multiplying and dividing by $a + b\sqrt{3}$

$= \underbrace{\frac{a}{a^2 - 3b}}_{\in \mathbb{Q}} - \underbrace{\frac{b}{a^2 - 3b}}_{\in \mathbb{Q}}\sqrt{3}$ $\Rightarrow$ $\frac{1}{x}$ is in our set. $\Rightarrow$ our set is a subgroup of $\mathbb{R}$

$\boxed{x^{-1} = \frac{1}{x}}$ (in the usual sense of division of numbers)

1) (b) The answer is $\boxed{\text{no}}$ The inverse of $x = a + b\sqrt{3}$ has to be $\frac{1}{x}$. Take $x = 1 + \sqrt{3}$ $\Rightarrow$ $\frac{1}{x} = \frac{1}{1+\sqrt{3}} = \frac{1 - \sqrt{3}}{1 - 3} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$

-you can continue your solution in the next page, if you do not have enough space

NAME ________________

Circle First Letter of Last Name

A-F G-K L-O P-Z

So, $\frac{1}{1-\sqrt{3}} = a + b\sqrt{3}$ but a and b are not integers

(Here it is worth to note that the representation of x as $a+b\sqrt{3}$ with $a, b \in \mathbb{Q}$, if exists, is unique.

Indeed, let $a_1 + b_1\sqrt{3} = a_2 + b_2\sqrt{3}$ and $b_1 \neq b_2 \Rightarrow$

$\sqrt{3} = \frac{a_1 - a_2}{b_2 - b_1} \in \mathbb{Q}$ but $\sqrt{3} \notin \mathbb{Q}$

$\Rightarrow b_1 = b_2 \Rightarrow a_1 = a_2$)

Hence $\frac{1}{1-\sqrt{3}}$ is not in our set $\Rightarrow$ $1+\sqrt{3}$ does not have an inverse in our set $\Rightarrow$ our set is not a group

c) The answer is $\boxed{\text{yes}}$: 0) The set is closed under the operation

1) associativity $(a*b)*c = \frac{1}{5}ab*c = \frac{1}{25}abc$ $\Rightarrow (a*b)*c = a*(b*c)$

$a*(b*c) = a*\frac{1}{5}bc = \frac{1}{25}abc$

2) existence of identity: e is the identity $\Leftrightarrow$ $e*a = a*e = a$ for any real $a \neq 0$ $\Leftrightarrow$ $\frac{1}{5}ea = a \Rightarrow \frac{e}{5} = 1 \Leftrightarrow \boxed{e=5}$

$e=5$ is the identity

3) existence of inverse for any $a \neq 0$: x is the inverse of a $\Leftrightarrow$ $a*x = x*a = 5 \Leftrightarrow \frac{1}{5}ax = 5 \Rightarrow x = \frac{25}{a} \Rightarrow$ the inverse existence and equal to $\boxed{a^{-1} = \frac{25}{a}}$

Problem 2 $bxab^{-1} = b^{-1}ab \Rightarrow$ (multiply by b^{-1} from the left and by $(ab^{-1})^{-1} = (b^{-1})^{-1}a^{-1} = ba^{-1}$ from the right)

$x = b^{-1}b^{-1}abba^{-1} = \boxed{b^{-2}ab^2a^{-1}}$