Extra credit regarding some steps in the proof of compatibility of a conification, MATH666, Fall 2013

submit before the end of last class of the term

1. Assume that $\lambda : [0,T] \mapsto [0,1]$ be an integrable function and for any natural n let A_n be a subset of $[0,T]$ defined as follows

$$A_n := \bigcup_{k=0}^{n-1} \left[\frac{Tk}{n}, \frac{Tk}{n} + \int_{\frac{Tk}{n}}^{\frac{T(k+1)}{n}} \lambda(t)\,dt\right].$$

Let χ_{A_n} be the characteristic function of the set A_n:

$$\chi_{A_n}(t) = \begin{cases} 1 & \text{if } t \in A_n, \\ 0 & \text{if } t \notin A_n \end{cases}$$

Prove that for any function $\varphi \in L_1([0,T])$ (i.e. such that $|\varphi|$ is integrable on $[0,T]$)

$$\int_0^T \chi_{A_n}(\tau)\varphi(\tau)\,d\tau \xrightarrow[n\to\infty]{} \int_0^T \lambda(\tau)\varphi(\tau)\,d\tau$$

and the convergence is uniform on $[0,T]$ (Hint: first prove the statement for a characteristic function $\chi_{[a,b]}$ of an interval and then use that the set of step functions are dense in $L_1([0,T])$).

2. Let Y_1 and Y_2 be two complete vector fields on $\mathbb{R}^n$, which are also Lipshitzian, i.e there exist $L_i > 0$ such that $||Y_i(x_1) - Y_i(x_2)|| \leq L_1||x_1 - x_2||$ for any $x_1, x_2 \in \mathbb{R}^n$, $i \in \{1,2\}$. Let T, λ, A_n be as in the previous problem. Further, assume that $q(t)$ is the trajectory of the vector field $\lambda(t)Y_1 + (1-\lambda(t))Y_2$ with $q(0) = q_0$ and $q_n(t)$ are the trajctories of the vector field $\chi_{A_n}(t)Y_1 + (1-\chi_{A_n}(t))Y_2$ with $q_n(0) = q_n$ and such that $q_n \xrightarrow[n\to\infty]{} q_0$. Prove that $q_n(t) \xrightarrow[n\to\infty]{} q(t)$ uniformly on $[0,T]$.

Hint: Use the Gronwall inequality: if $y(\tau)$, $\beta(\tau)$ are continuous and take non-negative valued on $[0,T]$, $\alpha \geq 0$ such that $y(t) \leq \alpha + \int_0^t \beta(\tau)y(\tau)\,d\tau$ for $t \in [0,T]$, then $y(t) \leq \alpha e^{\int_0^t \beta(\tau)\,d\tau}$.

Remark: In this setting some assumptions (like $M = \mathbb{R}^n$ instead of general manifold) are stronger than those I made in class and some assumptions are weaker (like $q_n \xrightarrow[n\to\infty]{} q_0$ instead of $q_n = q_0$). To use this statement for proving the compatibility of the convex combination we cover the trajectory $q(t)$ by finite number of coordinate neighborhoods (it is possible to do due compactness) and use the statement on each of them (we need to assume that $q_n \xrightarrow[n\to\infty]{} q_0$ instead of $q_n = q_0$ in order to make the gluing).