Lecture notes of Wednesday, Nov. 21

MATH 308-505 (summarizing also what we discussed previously)

Repeated eigenvalues continued

Prop. 1 Let $v_1, \dots, v_n$ be a basis in $\mathbb{R}^n$

($\iff \det(v_1, \dots, v_n) \neq 0 \iff$ for any vector v there exists unique $\alpha_1, \dots, \alpha_n$ such that $v = \alpha_1 v_1 + \dots + \alpha_n v_n$). Assume that

$$Av_j = \sum_{i=1}^{n} b_{ij} v_i \qquad (*)$$

$$\left.\begin{aligned} B &= (b_{ij})_{i,j=1}^{n} \\ Q &= (v_1, \dots, v_n) \end{aligned}\right\} \text{two } n \times n \text{ matrices.}$$

Then $\boxed{Qe^{tB}}$ is a fundamental matrix of the system $x' = Ax$

Explanation Formula $(*)$ can be written in matrix form as $AQ = QB$ or equivalently $A = QBQ^{-1}$, where Q^{-1} is an inverse matrix of Q ($QQ^{-1} = Q^{-1}Q = I$)

As proved in the previous class e^{tA} is a fundamental matrix of the system $x' = Ax$

Calculate e^{tA} in terms of B and Q

<u>Lemma</u> $\boxed{e^{tA} = Qe^{tB}Q^{-1}}$

<u>Proof</u> $e^{tA} = I + tA + \frac{t^2}{2}A^2 + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$

$I = QQ^{-1}$

$A = QBQ^{-1}$

$A^2 = (QBQ^{-1})^2 = (QBQ^{-1})(QBQ^{-1}) = QB\underbrace{Q^{-1}Q}_{I}BQ^{-1} =$

$= QB^2Q^{-1}$

$A^3 = (QBQ^{-1})^3 = QB\underbrace{Q^{-1}Q}_{I}B\underbrace{Q^{-1}Q}_{I}BQ^{-1} = QB^3Q^{-1}$

In general

$A^i = QB^iQ^{-1} \Rightarrow$

$e^{tA} = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!} = \sum_{i=0}^{\infty} \frac{t^i QB^iQ^{-1}}{i!} = Q \underbrace{\sum_{i=0}^{\infty} \frac{t^i B^i}{i!}}_{e^{tB}} Q^{-1}$

$= Qe^{tB}Q^{-1} \Rightarrow$ Proof of lemma is completed. $\square$

So, $Qe^{tB}Q^{-1}$ is a fundamental matrix

$\left(Q e^{tB} Q^{-1}\right) Q = Q e^{tB}$ is a fundamental matrix of $x' = Ax$ $\Rightarrow$

<u>Rem</u> In general if $\Psi(t)$ is a fundamental matrix then $\Psi(t)\, C$ is a fundamental matrix for any $n \times n$ matrix C with $\det C \neq 0$.

Based on Prop. 1 our goal is to find a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that the corresponding matrix B will be as simple as possible (so that we will be able to calculate e^{Bt})

Example 1

<u>The most simple case is</u> when there exists a basis $v_1, \dots, v_n$ consisting of the eigenvectors of A. Indeed if $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues, i.e

$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ &\vdots \\ Av_n &= \lambda_n v_n \end{aligned}$$

Then the corresponding matrix B is a diagonal matrix

$$B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$ (in this case A is called diagonizable)

and $e^{Bt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & e^{\lambda_n t} \end{pmatrix} \overset{\text{Prop 1}}{\Longrightarrow}$

$$\underbrace{\begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}}_{Q} \begin{pmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & & \\ 0 & \dots & 0 & e^{\lambda_n t} \end{pmatrix} =$$

$= \begin{pmatrix} e^{\lambda_1 t} v_1 & e^{\lambda_2 t} v_2 & \dots & e^{\lambda_n t} v_n \end{pmatrix}$ is

a fundamental matrix of $x' = Ax$, which recovers what we already previously did without matrix exponential.

However not any matrix A is diagonizable ($\Leftrightarrow$ not always there is a basis of eigenvectors of A)

Example 2

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

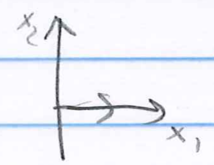
$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 = 0 \Rightarrow$$

there exists a unique eigenvalue $\lambda = 2$ (repeated)

Find the eigenvectors of $\lambda = 2$

$(A - 2I)v = 0 \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_2 = 0 \Rightarrow$

$v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ is an eigenvector



Eigenvectors span a line (x_1-axis) but not the whole R^2

$\Rightarrow$ there is no a basis of eigenvectors.

Algebraic and geometric multiplicity of an eigenvalue

1) Algebraic multiplicity.

$\lambda = \lambda_1$ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda_1 I) = 0 \Leftrightarrow$

$\lambda = \lambda_1$ is a root of characteristic polynomial $\det(A - \lambda I) = 0$

Def An algebraic multiplicity of ~~of~~ λ_1 is the multiplicity of the root $\lambda = \lambda_1$ in the characteristic polynomial $\det(A - \lambda I)$, i.e. the maximal power of $(\lambda - \lambda_1)$ in the factorization of $\det(A - \lambda I)$ in linear factors

Recall If, for example, $P(\lambda) = (\lambda-2)^3(\lambda-3)^2(\lambda-1)$

then a root 2 has multiplicity 3, a root 3

has multiplicity 1, and a root 1 has multiplicity 1 in P (as we discussed in the method of undetermined coefficients)

2) Geometric multiplicity

$\lambda = \lambda_1$ is an eigenvalue $\Leftrightarrow$ there exists nonzero vector v s.t. $Av = \lambda_1 v \Leftrightarrow$ the linear system

$(A - \lambda_1 I)v = 0$ (**) has non-zero solutions

Let E_{λ_1} be a set of all solutions of (**).

Here we have a superposition principle:

If v_1 and v_2 are solutions of (**) then $c_1 v_1 + c_2 v_2$ are solutions of (**) for any constants c_1 and c_2. So E_{λ_1} is a vector subspace of R^n (it could be a line, a plane, a 3-dim subspace etc)

E_{λ_1} is called the eigenspace of λ_1.

Def Geometric multiplicity of λ_1 is the dimension of E_{λ_1} (i.e the maximal number of linearly independent vectors in E_{λ_1})

Prop 2 Geometric multiplicity $\leq$ Algebraic multiplicity

(A proof is omitted, but it is not difficult)

Thm A matrix is diagonizable $\Leftrightarrow$ there exists a basis of eigenvectors) $\Leftrightarrow$ for any its eigenvector the geometric multiplicity is equal to the algebraic multiplicity.

(A proof is omitted)

Continuation of Example 2

In Example 2 $\lambda = 2$ is the eigenvalue of algebraic multiplicity 2, because the characteristic polynomial is $(\lambda-2)^2$, but its geometric multiplicity is equal to 1, because the eigenspace $E_2 = \left\{ \begin{pmatrix} v_1 \\ 0 \end{pmatrix} ; v_1 \in K \right\}$, i.e. $\dim E_2 = 1$.

What to do if A is not diagonizable

Case $n=2$. If λ_1 is a repeated root then its algebraic multiplicity $=2$

2 possibilities

Case 1 Geom. multiplicity of λ_1 is equal to 2

$\Leftrightarrow (A - \lambda_1 I) v = 0$ for any $v \in \mathbb{R}^2 \Leftrightarrow$

$$A = \lambda_1 I = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \text{ i.e. } A \text{ is diagonal}$$

so we know how to treat this case

$$e^{tA} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_1} \end{pmatrix}$$

Case 2 Geom. multiplicity of λ_1 is equal to 1

$\Leftrightarrow \dim E_{\lambda_1} = 1$ ($<$ alg. multiplicity)

How do solve the system of differential equations in this case?

The algorithm:

<u>Step 1</u> Find an eigenvector v of λ_1 (it is defined up to a multiple) by solving the system

$$(A-\lambda_1 I)v = 0 \quad (\text{as usual})$$

<u>Step 2</u> Once v is found solve an additional system

$$(A-\lambda_1 I)w = v \qquad (***)$$

for w

(Then $(A-\lambda_1 I)^2 w = (A-\lambda_1 I)\underbrace{(A-\lambda_1 I)w}_{} =$
$= (A-\lambda_1 I)v = 0$. Such w is called the <u>generalized eigenvector</u> (of order 2) in this case)

Note that the solution of $(***)$ indeed exists. Let us explain this.

The set $\{(A-\lambda_1 I)w,\ w \in \mathbb{R}^2\}$ is not the whole $\mathbb{R}^2$ (because $\det(A-\lambda_1 I)=0$) and it is not equal to 0 (because $A-\lambda_1 I \neq 0$). Therefore it is a line. Let us show that this line is generated by the eigenvector v from the step 1. Indeed if it is generated by some w_0 then in particular $(A-\lambda_1 I)w_0 = cw_0 \Rightarrow$ w_0 is an eigenvector of A $\left((A-(\lambda_1-c))w_0 = 0\right)$ but we know that eigenvectors of A are all multiples of $v \Rightarrow w_0 = c_1 v$ $\blacksquare$

Rem Note also that the solution of $(\times\times\times)$ is not unique: If $(A-\lambda_1 I)w = v$ then $(A-\lambda_1 I)(v+cv) = 0$ for any c.

In any case v and w from Step 1 and Step 2 constitute a basis of R^2

(w cannot be a multiple of v because otherwise $(A-\lambda_1 I)w = 0 = v$, but $v \neq 0$)

Now find the matrix B from Prop 1 (of the very first page) corresponding to A in the basis v and w (looking at v as v_1 and w as v_2)

$Av = \lambda_1 v$

$(A-\lambda_1 I)w = v \Leftrightarrow Aw - \lambda_1 w = v \Leftrightarrow$

$Aw = v + \lambda_1 w \Rightarrow$ the matrix B has the form $B = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \to$ <u>Jordan block</u> of size 2

Prop 1 $\Rightarrow$ The matrix $(v\ w)\, e^{Bt}$ is a fundamental matrix of $x' = Ax$

Let us find e^{tB}

$$B = \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}}_{\substack{D \\ \underbrace{}_{\text{diagonal part}}}} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\substack{N \\ \underbrace{}_{\text{nilpotent part}}}}$$

Rem In general given two matrices L_1 and L_2 $e^{L_1+L_2} \neq e^{L_1}e^{L_2}$

(the reason for this is that in general $L_1L_2 \neq L_2L_1$, i.e. L_1 and L_2 do not commute)

However, if $L_1L_2 = L_2L_1$ then $e^{L_1+L_2} = e^{L_1}e^{L_2}$

In our case $L_1 = D$, $L_2 = N$ and D commutes with any matrix $\Rightarrow$

$$e^{tB} = e^{tD}e^{tN} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{pmatrix} e^{tN}$$

Let us calculate e^{tN}

$$e^{tN} = I + Nt + \frac{N^2 t^2}{2} + \frac{N^3 t^3}{3!} + \dots$$

Note that $N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

so $N^3 = 0$ and $N^i = 0$ for $i \geq 2 \Rightarrow$

$$e^{tN} = I + Nt = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$\Downarrow$

$$e^{tB} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} v & w \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} v & te^{\lambda_1 t} v + e^{\lambda_1 t} w \end{pmatrix}$$

is the fundamental matrix of $x' = Ax \Rightarrow$

$$\boxed{\begin{aligned} &c_1 e^{\lambda_1 t} v + c_2 (te^{\lambda_1 t} v + e^{\lambda_1 t} w) = \\ &= (c_1 + c_2 t) e^{\lambda_1 t} v + c_2 e^{\lambda_1 t} w \end{aligned}}$$

is <u>the general solution</u> of $x' = Ax$