Main properties of Laplace transform

Definition 1. *(piecewise continuity) A function f is called piecewise continuous on the interval $a \leq t \leq b$ if there exists a partition of this interval by finite number of points $\alpha = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ such that*

(1) *f is continuous on each open subinterval $t_{i-1} < t < t_i$,*
(2) *one-sided limits $\lim\limits_{t \to t_i+} f(t)$ and $\lim\limits_{t \to t_i-} f(t)$ exist and finite.*

A function f is called piecewise continuous on $[0, +\infty]$ if it is piecewise continuous on the interval $[0, A]$ for any $A > 0$.

Examples:

Definition 2. *(functions of exponential order a) A function f is said to be of exponential order a as $t \to +\infty$ if there exist real constants $M \geq 0$, $K > 0$ and a such that*

$$|f(t)| \leq Ke^{at}, \quad \text{for all } t \geq M.$$

Examples:

Theorem 1. *(on existence of Laplace transform) If $f(t)$ is piecewise continuous on $[0, +\infty]$ and of the exponential order a then the Laplace transform $F(s)$ of $f(t)$ exists for any $s > a$. Moreover, $|F(s)| \leq \frac{L}{s}$ for some positive constant L.*

Sketch of the proof:

Main properties (regarding problems of section 6.2):

(1) Translation in s:
$$\mathcal{L}\left\{e^{\alpha t}f(t)\right\} = F(s-\alpha);$$

Proof.

$\mathcal{L}\left\{e^{\alpha t}\sin\beta t\right\} =$

$\mathcal{L}\left\{e^{\alpha t}\cos\beta t\right\} =$

(2) Laplace transform of the derivative:
$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$$

More generally,
$$\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0),$$

By induction,
$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0);$$

(3) Derivative of Laplace transform:
$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n F^{(n)}(s).$$

Example $\mathcal{L}\left\{t^n e^{\alpha t}\right\} =$

Theorem 2. *(Existence of the inverse Laplace transform) If f and g are piecewise continuous functions of exponential order a on $[0,+\infty)$ and they have the same Laplace transform, i.e. $\mathcal{L}\left\{f(t)\right\} \equiv \mathcal{L}\left\{g(t)\right\}$, then $f(t) = g(t)$ at all points of continuity of the functions f and g. In particular, if f and g are continuous then $f \equiv g$.*