So we will get (for each chain $w^1, w^2, \dots, w^k$ in the basis of generalized eigenvectors) that the chain gives k independent solutions

$$e^{tA}w^1 = e^{\lambda t}\left(w^1 + tw^2 + \frac{t^2}{2}w^3 + \dots + \frac{t^{k-1}}{(k-1)!}w^k\right)$$

$$e^{tA}w^2 = e^{\lambda t}\left(w^2 + tw^3 + \frac{t^2}{2}w^4 + \dots \frac{t^{k-2}}{(k-2)!}w^k\right)$$

$$\vdots$$

$$e^{tA}w^{k-1} = e^{\lambda t}\left(w^{k-1} + tw^k\right)$$

$$e^{tA}w^k = e^{\lambda t}w^k$$

<u>Remark</u> If we denote by $p(t) = w^1 + tw^3 + \frac{t^2}{2}w^3 + \dots + \frac{t^{k-1}}{(k-1)!}w^k$,

then $e^{tA}w^1 = e^{\lambda t}p(t)$, $e^{tA}w^2 = e^{\lambda t}\frac{d}{dt}p(t)$,

$e^{tA}w^3 = e^{\lambda t}\frac{d^2}{dt^2}p(t), \dots, e^{tA}w^k = e^{\lambda t}\frac{d^{k-1}}{dt^{k-1}}p(t)$

Of course, it is some work to find the basis of generalized eigenvectors consisting of such chains

• For example if $n=2$, repeated root λ of geometric multiplicity 1 we take a generalized eigenvector w of order 2 (the highest possible order) which is not an eigenvector and form the chain: w, $v=(A-\lambda I)w$ of length 2.

Then
$$e^{tA}w = e^{\lambda t}(w+tv)$$
$$e^{tA}v = e^{\lambda t}v$$

Note that $p(t)$ here is $w+tv$ and $p'(t)=v$

is the fundamendal set of solutions

• If $n=3$ and there is one repeated eigenvalue λ_1 of algebraic multiplicity 2 and another eigenvalue λ_2 (inevidebly of multiplicity 1) (as in problem 3 of hwk 15), then similar to the previous case for λ_1 take a generalized eigenvector w of order 2 (the highest possible order for λ_1) and form a chain w, $v^1=(A-\lambda_1 I)w$, of length 2. Then exactly as in the previous case

$$e^{tA} w = e^{\lambda_1 t}(w + t v^1)$$

$$e^{tA} v^1 = e^{\lambda_1 t} v^1$$

For λ_2: the highest possible order of generalized eigenvectors is 1 (i.e. it is an eigenvector)
Take such eigenvector v^2 and this vector itself form a chain (of length 1) so that

$$e^{tA} v^2 = e^{\lambda_2 t} v^2$$

Hence all obtained 3 solutions form a fundamental set of solutions

- If $n=3$ and there is one eigenvalue λ of algebraic multiplicity 3 and geometric multiplicity 1 (as in the bonus problem 3a of homework #16) then similarly to the previous cases take a generalized eigenvector w of order 3 (the highest possible order) which is not of order 2 and form the chain w, $w' = (A-\lambda I)w$, $v = (A-\lambda I)^2 w$ of length 3. Then

$$e^{tA}w = e^{\lambda t}\left(w + tw' + \frac{t^2}{2}v\right)$$

$$e^{tA}w' = e^{\lambda t}(w' + tv)$$

$$e^{tA}v = e^{\lambda t}v$$

is a fundamental set of solutions (where

$$p(t) = w + tw' + \frac{t^2}{2}v,\ p'(t) = w' + tv,\ p''(t) = v)$$

. If $n=3$ and there is one repeated eigenvalue of algebraic multiplicity 3 and geometric multiplicity 2 (as in the bonus problem 3b of homework #17) then similarly to the previous cases take a generalized eigenvector w of order 2 (the highest possible order in this case) which is not an eigenvector and form the chain w, $v' = (A-\lambda I)w$ of length 2

Then as in the first two cases

$$e^{tA}w = e^{\lambda t}(w + t v^1)$$

$$e^{tA}v^1 = e^{\lambda t}v^1$$

In general here we may take any w^1 which complete v^1, w to the basis in $\mathbb{R}^3$. Then w^1 is a generalized eigenvector of order at most 2. So it would be easy to calculate $e^{\lambda t}w^1$ to get a third independent solution. However, since the eigenspace of λ is 2-dimensional and we used so far only one eigenvector v^1 it is more convenient here to choose another independent eigenvector v^2. This eigenvector itself form a chain (of length 1) and

$$e^{tA}v^2 = e^{\lambda t}v^2$$

The obtained 3 solutions form a fundamental set of solutions.

In general, in order to understand what is the highest order of generalized eigenvector for a given eigenvector λ you must ~~to~~ study system of equations

$$(A-\lambda I)v=0, \quad (A-\lambda I)^2 v=0, \quad (A-\lambda I)^3=0 \text{ etc}$$

(until the space of solution of

$(A-\lambda I)^{k+1} v = 0$ coincides with $(A-\lambda I)^k v=0$)

Hopefully, from this description you will be able to understand how to form the chains of generalized eigenvectors in general situation and therefore solve any linear system of ordinary differential equations with constant coefficients.