-1- Homework # 1 Solutions (Regular section part) Math 303

Problem 1 $Q' = -rQ \Rightarrow Q(t) = Q(0)e^{-rt}$ (as shown in class).

$\Rightarrow$ if t_1 is the half-life then $\frac{Q(t_1)}{Q(0)} = \frac{1}{2} \Rightarrow$

$$\frac{\cancel{Q(0)}e^{-rt_1}}{\cancel{Q(0)}} = \frac{1}{2} \Rightarrow e^{-rt_1} = \frac{1}{2} \Rightarrow -rt_1 = \ln\frac{1}{2} = -\ln 2 \Rightarrow$$

$$\Rightarrow \boxed{t_1 = \frac{\ln 2}{r}}$$

Problem 2 **1a)** Assume that $v(t) \equiv v_e$ is a solution. Plugging into equation (1): $v'(t) = 0 = 9.8 - \frac{v_e}{a} \Rightarrow \boxed{v_e = 9.8a}$

(1b) $v' = 9.8 - \frac{v}{a} \underset{\text{separating}}{\Rightarrow} \frac{dv}{9.8 - \frac{v}{a}} = dt \underset{\text{integrate}}{\Rightarrow} \int \frac{dv}{9.8 - \frac{v}{a}} = t + C_1$

$$\int \frac{dv}{9.8 - \frac{v}{a}} = -a\int \frac{du}{u} = -a\ln|u| = -a\ln\left|9.8 - \frac{v}{a}\right| \Rightarrow$$

$u = 9.8 - \frac{v}{a} \Rightarrow du = -\frac{1}{a}dv \Rightarrow dv = -a\,du$

$$-a\ln\left|9.8 - \frac{v}{a}\right| = t + C_1 \Leftrightarrow \ln\left|9.8 - \frac{v}{a}\right| = -\frac{t}{a} + \underbrace{\left(-\frac{C_1}{a}\right)}_{\text{call it } C_2}$$

Exponentiate $\Rightarrow$

$$\left|9.8 - \frac{v}{a}\right| = e^{C_2}e^{-t/a} \Rightarrow 9.8 - \frac{v}{a} = \underbrace{(\pm e^{C_2})}_{\text{call it } C_3}e^{-t/a} \Leftrightarrow$$

$$9.8 - \frac{v}{a} = C_3e^{-t/a} \Rightarrow \frac{v}{a} = 9.8 - C_3e^{-t/a} \Rightarrow$$

$$v(t) = 9.8a \underbrace{-aC_3}_{\text{call it } C} e^{-t/a} = 9.8a + Ce^{-t/a} \quad (1)$$

Here you also could use class notes:

$y' = \alpha y + \beta$ has a general solution $y(t) = -\frac{\beta}{\alpha} + Ce^{\alpha t}$ (2)

In ou[illegible] $\alpha = -\frac{1}{a}$ and $\beta = 9.8$ [illegible]

2- In our case $\alpha = -\frac{1}{a}$ and $\beta = 9.8 \Rightarrow -\frac{\beta}{\alpha} = 9.8a$ and we get (1)
from 2

(1) & initial conditions $v(0) = 4.9a \Rightarrow$

$$4.9a = 9.8a + C \Rightarrow C = -4.9a \Rightarrow$$

$$\boxed{v(t) = 9.8a - 4.9a\,e^{-t/a} = 4.9a\left(2 - e^{-t/a}\right)}$$

(c) If t_1 is required time then

$$v(t_1) = \frac{9}{10} \cdot \cancel{9.8}^{2}\cancel{a} = \cancel{4.9a}\left(2 - e^{-t_1/a}\right) \Rightarrow$$

$$\frac{9}{5} = 2 - e^{-t_1/a} \Rightarrow e^{-t_1/a} = 2 - \frac{9}{5} = \frac{1}{5} \Rightarrow$$

$$-t_1/a = \ln\frac{1}{5} = -\ln 5 \Rightarrow t_1 = (\ln 5)a$$

(d) the distance $= \int_0^{\ln 5 a} 4.9a\left(2 - e^{-t/a}\right) dt = 4.9a\left(2t + a e^{-t/a}\right)\Big|_0^{(\ln 5)a} =$

$$= 4.9a\left(2\ln 5\, a + a \underbrace{e^{-\frac{a\ln 5}{a}}}_{e^{-\ln 5} = \frac{1}{5}} - a\right) = \boxed{4.9a^2\left(2\ln 5 - \frac{4}{5}\right)}$$

Problem 3

(a) $2t + ty^2 + e^{t^2} y y' = 0$

Express y': $e^{t^2} y y' = -2t - ty^2 = -t(y^2+2) \Rightarrow$

$$y' = \underbrace{-\frac{t}{e^{t^2}} \frac{y^2+2}{y}}_{\text{seperable}}$$

Separate $\Rightarrow$ $\frac{y\,dy}{y^2+2} = -t e^{-t^2} dt \xRightarrow{\text{Integrate}} \int \frac{y\,dy}{y^2+2} = \int -t e^{-t^2} dt$

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$$\int \frac{y}{y^2+2}\,dy = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(y^2+2)$$

u-substitution
$u = y^2+2 \Rightarrow du = 2dy \Rightarrow dy = \frac{1}{2}du$

$$\int -te^{-t^2}\,dt = \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{-t^2}$$

u-substitution $u = -t^2 \Rightarrow du = -2t\,dt \Rightarrow -t\,dt = \frac{1}{2}du$

$\Downarrow$

$$\cancel{\frac{1}{2}}\ln(y^2+2) = \cancel{\frac{1}{2}}e^{-t^2} + C_1$$

Exponentiate: $y^2+2 = e^{C_1} e^{e^{-t^2}} \Rightarrow y^2+2 = Ce^{e^{-t^2}} \Rightarrow$

$e^{C_1} = C > 0$

$$y^2 = Ce^{e^{-t^2}} - 2 \Rightarrow \boxed{y = \pm\sqrt{Ce^{e^{-t^2}} - 2}},\quad C > 0$$

b) $(xy^2+x)\frac{dy}{dx} + x^2y - y = 0$

Express $\frac{dy}{dx}$: $\underbrace{(xy^2+x)}_{x(y^2+1)}\frac{dy}{dx} = \underbrace{y - x^2y}_{y(1-x^2)} \Rightarrow \frac{dy}{dx} = \underbrace{\frac{y}{y^2+1}\cdot\frac{1-x^2}{x}}_{\text{separable}}$

separate $\Rightarrow$ $\frac{y^2+1}{y}\,dy = \frac{1-x^2}{x}\,dx$ or $\left(y+\frac{1}{y}\right)dy = \left(\frac{1}{x}-x\right)dy$

Integrate: $\int\left(y+\frac{1}{y}\right)dy = \int\left(\frac{1}{x}-x\right)dx + C \Rightarrow$

$$\boxed{\frac{y^2}{2} + \ln|y| = \ln|x| - \frac{x^2}{2} + C} \rightarrow \text{general solution}$$

It was a misprint in the original text of the assignment regarding the initial conditions: $y(0)=e$ is impossible. This part will not be graded, But I meant $y(1)=e$.

I still want to give a solution:

Plugging $y(1)=e$ into $\frac{y^2}{2}+\ln|y| = \ln|x| - \frac{x^2}{2} + C$:

$$\frac{e^2}{2} + \underbrace{\ln e}_{1} = \underbrace{\ln 1}_{0} - \frac{1}{2} + C \Rightarrow C = \frac{e^2}{2} + \frac{3}{2}$$

$$\Rightarrow \boxed{\frac{y^2}{2} + \ln|y| = \ln|x| - \frac{x^2}{2} + \frac{e^2}{2} + \frac{3}{2}}$$

Problem 4 (a) $y' = \frac{y-2x}{2x+y} = \frac{\frac{y}{x}-2}{2+\frac{y}{x}}$

Let $u = \frac{y(x)}{x} \Rightarrow y(x) = xu(x) \Rightarrow$
$y' = xu' + u \Rightarrow$ substituting u instead of y into the original equation we get

$$xu' + u = \frac{u-2}{2+u} \Leftrightarrow xu' = \frac{u-2}{u+2} - u = \frac{u-2-u^2-2u}{u+2} =$$

$$= -\frac{u^2+u+2}{u+2} \Rightarrow xu' = -\frac{u^2+u+2}{u+2} \overset{\text{Separate}}{\Rightarrow}$$

$$\frac{(u+2)\,du}{u^2+u+2} = -\frac{dx}{x}$$

Integrate:
$$\int \frac{(u+2)\,du}{u^2+u+2} = -\int \frac{dx}{x} + C$$

$$\int \frac{u+2}{u^2+u+2}\,du = \frac{1}{2}\int \frac{2u+1}{u^2+u+2}\,du + \frac{5}{2}\int \frac{du}{(u+\frac{1}{2})^2+\frac{7}{4}} =$$

$$\frac{u+2}{u^2+u+2} = \frac{1}{2}\underbrace{\frac{2u+1}{u^2+u+2}}_{\text{logarithmic derivative}} + \frac{3}{2}\,\frac{1}{\underbrace{u^2+u+2}_{(u-\frac{1}{2})^2+\frac{7}{4}}}$$

$$= \frac{1}{2}\ln|u^2+u+2| + \cancel{\frac{3}{2}}\,\cancel{\frac{2}{\sqrt{7}}}\ \text{arctan}\,\frac{u+\frac{1}{2}}{\frac{\sqrt{7}}{2}} =$$

$$= \frac{1}{2}\ln|u^2+u+2| + \frac{3}{\sqrt{7}}\ \text{arctan}\,\frac{2u+1}{\sqrt{7}}$$ (here I use a table integral $\int \frac{1}{x^2+a^2} = \frac{1}{a}\,\text{arctan}\,\frac{x}{a}$)

$$\int \frac{dx}{x} = \ln|x| -$$

$$\Rightarrow\ \frac{1}{2}\ln|u^2+u+2| + \frac{3}{\sqrt{7}}\ \text{arctan}\,\frac{2u+1}{\sqrt{7}} = -\ln|x| + C$$

Returning to the original unknown function y:

Since $u=\frac{y}{x}$:

$$\boxed{\frac{1}{2}\ln\left|\frac{y^2}{x^2}+\frac{y}{x}+2\right| + \frac{3}{\sqrt{7}}\ \text{arctan}\,\underbrace{\frac{2\frac{y}{x}+1}{\sqrt{7}}}_{\frac{2y+x}{\sqrt{7}x}} = -\ln|x| + C}$$

4(b) $y' = (3x-2y-1)^2$

Let $u(x) = 3x-2y-1 \Rightarrow y(x) = \frac{1}{2}(3x-1-u(x)) \Rightarrow$

$y' = \frac{1}{2}(3-u') \Rightarrow$ substituting into the equation u instead of y:

$\frac{1}{2}(3-u') = u^2 \Leftrightarrow u' = 3-2u^2 \overset{\text{seperate}}{\Longrightarrow}$

$\frac{du}{3-2u^2} = dx \overset{\text{Integrate}}{\Longrightarrow} \int \frac{du}{3-2u^2} = x + C_1 \quad (*)$

$$\frac{1}{3-2u^2} = \frac{1}{(\sqrt{3}-\sqrt{2}u)(\sqrt{3}+\sqrt{2}u)} \overset{\text{partial fraction decomposition}}{=} \frac{A}{\sqrt{3}-\sqrt{2}u} + \frac{B}{\sqrt{3}+\sqrt{2}u} \Rightarrow$$

$1 = A(\sqrt{3}+\sqrt{2}u) + B(\sqrt{3}-\sqrt{2}u)$

If $u = -\sqrt{\frac{3}{2}} \Rightarrow 1 = B\left(\sqrt{3} - \sqrt{2}\times\left(-\frac{\sqrt{3}}{\sqrt{2}}\right)\right) = 2\sqrt{3}B \Rightarrow$

$B = \frac{1}{2\sqrt{3}}$

If $u = \sqrt{\frac{3}{2}} \Rightarrow 1 = 2\sqrt{3}A \Rightarrow A = \frac{1}{2\sqrt{3}} \Rightarrow$

$$\frac{1}{3-2u^2} = \frac{1}{2\sqrt{3}}\left(\frac{1}{\sqrt{3}-\sqrt{2}u} + \frac{1}{\sqrt{3}+\sqrt{2}u}\right) \Rightarrow$$

$$\int \frac{du}{3-2u^2} = \frac{1}{2\sqrt{3}}\left(\int \frac{du}{\sqrt{3}-\sqrt{2}u} + \int \frac{du}{\sqrt{3}+\sqrt{2}u}\right) = \frac{1}{2\sqrt{6}}\left(-\ln\left|\sqrt{3}-\sqrt{2}u\right| + \ln\left|\sqrt{3}+\sqrt{2}u\right|\right) = \frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3}-\sqrt{2}u}\right|$$

Returning to the equation (*) on page 6

$$\frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3}-\sqrt{2}u}\right| = x + C_1 =$$

$$\ln\left|\frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3}-\sqrt{2}u}\right| = 2\sqrt{6}x + \underbrace{2\sqrt{6}C_1}_{C_2}$$

Exponentiate: $\left|\frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3}-\sqrt{2}u}\right| = e^{C_2} e^{2\sqrt{6}x}$

$$\frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3}-\sqrt{2}u} = \underbrace{\pm e^{C_2}}_{C} e^{2\sqrt{6}x} = Ce^{2\sqrt{6}x}$$

Express u in terms of x (this is possible in this case:

$$\sqrt{3}+\sqrt{2}u = \left(\sqrt{3}-\sqrt{2}u\right)Ce^{2\sqrt{6}x} \Rightarrow$$

$$\left(\sqrt{2}+\sqrt{2}Ce^{2\sqrt{6}x}\right)u = \sqrt{3}Ce^{2\sqrt{6}x} - \sqrt{3} \Rightarrow$$

$$u = \frac{\sqrt{3}}{\sqrt{2}} \frac{Ce^{2\sqrt{6}x}-1}{Ce^{2\sqrt{6}x}+1}$$

Returning to the original unknown function:

$$y(x) = \frac{1}{2}(3x-1-u) = \boxed{\frac{1}{2}\left(3x-1-\frac{\sqrt{3}}{\sqrt{2}} \frac{Ce^{2\sqrt{6}x}-1}{Ce^{2\sqrt{6}x}+1}\right)}$$