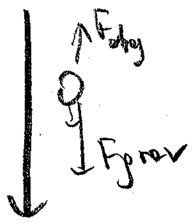
Homework 1 — Solution of honors section part
MATH 308

Problem 2 (e) According to the second Newton's law

$$m\vec{a} = \vec{F}_{grav} + \vec{F}_{drag} \iff \boxed{mv' = mg - \gamma v^2}$$



positive direction down

$F_{drag} = -\gamma v^2$ if the object falls (i.e. $v>0$)

If objects rises up then $F_{drag} = \gamma v^2$

So more right to write

$$\boxed{mv' = mg - \gamma \underbrace{(\operatorname{sign} v)}_{\text{the sign of } v} v^2}$$

Problem 5 (a) $\dfrac{dy}{dt} = \dfrac{ay+bt+m}{cy+dt+n}$

this are 2 different things (sorry for ambiguity)

$t = T+h \Rightarrow dt = dT$

$y = Y+k \Rightarrow dy = dY$

$$\Rightarrow \frac{dY}{dT} = \frac{dy}{dt} = \frac{a(Y-k)+b(T-h)+m}{c(Y-k)+d(T-h)+n} =$$

$$= \frac{aY+bT+(m-ak-bh)}{cY+dT+(n-ck-dh)}$$

We want to find k and h such that

$$\begin{matrix} m-ak-bh=0 \\ n-ck-dh=0 \end{matrix} \iff \begin{cases} ak+bh=m \\ ck+dh=n \end{cases} \quad (*)$$

-2- The condition $ad-bc\neq 0 \iff \det\begin{pmatrix} a & b\\ c & d\end{pmatrix}\neq 0$

$\iff$ geometrically means that the straight lines

$ak+bh=m$ & $ck+dh=m$ in (k,h)-plane

are not parallel $\Rightarrow$ they intersect $\Rightarrow$ there is a solution

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Remark: we will discuss in detail later in the course that $\det\begin{pmatrix} a & b\\ c & d\end{pmatrix}\neq 0$ implies that the system of page (1) has a solution (actually the unique one) for any m & n. You can show it by elimination or substitution to get that

$$(ad-bc)k = md-nc$$

$$(ad-bc)h = an-mc$$

So, $ad-bc\neq 0 \Rightarrow k=\dfrac{md-nc}{ad-bc}=\dfrac{\begin{vmatrix} m & c\\ n & d\end{vmatrix}}{\begin{vmatrix} a & b\\ c & d\end{vmatrix}}$ → Cramer

$h=\dfrac{an-mc}{ad-bc}=\dfrac{\begin{vmatrix} a & m\\ c & n\end{vmatrix}}{\begin{vmatrix} a & b\\ c & d\end{vmatrix}}$ → rule.

(b) $y'=\dfrac{y-2x+1}{2x+y-3}$, i.e. $\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 1 & -2\\ 1 & 2\end{pmatrix}$, $\begin{pmatrix} m\\ n\end{pmatrix}=\begin{pmatrix} 1\\ -3\end{pmatrix}$

Make change of variables $x=X+h$
$y=Y+k$

According to the previous item to arrive to the equation

$$\frac{dY}{dX} = \frac{aY+bX}{cY+dX} = \frac{Y-2X}{2X+Y} \quad (*)$$

we need to solve the system

$$\begin{cases} k-2h = 1 & (Eq\ 1) \\ k+2h = -3 & (Eq\ 2) \end{cases}$$

Elimination of k: (Eq 1)-(Eq 2) : $-4h = 4 \Rightarrow h = -1$

Elimination of h : (Eq 1)+(Eq 2) : $2k = -2 \Rightarrow k = -1$

So the desired coordinate change is

$$\begin{cases} x = X-1 \\ y = Y-1 \end{cases} \Leftrightarrow \begin{cases} X = x+1 \\ Y = y+1 \end{cases} \quad (**)$$

Eq (*) above is up to the notation, is the same as equation in 4(b) (just $Y \to y$, $X \to x$) so

from the solution of 4(b) the general solution of (*) is

$$\frac{1}{2}\ln\left|\frac{Y^2}{X^2}+\frac{Y}{X}+2\right| + \frac{3}{\sqrt{7}}\arctan\left(\frac{2Y+X}{\sqrt{7}X}\right) = -\ln|X| + C$$

Returning to the original coordinates using (**) we get

$$\boxed{\frac{1}{2}\ln\left|\frac{(y+1)^2}{(x+1)^2}+\frac{y+1}{x+1}+2\right| + \frac{3}{\sqrt{7}}\arctan\frac{2y+x+3}{\sqrt{7}x} = -\ln|x+1| + C}$$

Problem 5 c

If $ad-bc=0$ then there exist a number k

s.t. $a=kc$ & $b=kd$ (*)

(Indeed $ad-bc=0 \Leftrightarrow ad=bc \Rightarrow \frac{a}{c}=\frac{b}{d}$ if c & d are not $0 \Rightarrow$ take $k=\frac{a}{c}=\frac{b}{d}$. If $c=0$ then either a or $d=0$ and we k with property (*))

So the equation (2) is of the form

$$y'=\frac{k(cy+dt)+m}{cy+dt+n}=\frac{k(\cancel{cy+dt+n})}{\cancel{cy+dt+n}}+\frac{m-kn}{cy+dt+n}=$$

$$=k+\frac{m-kn}{cy+dt+n}=k+\frac{A}{cy+dt+n}, \text{ where } A=m-kn$$

This equation can be solved by the method of problem 4(b) (see also type 2 of page 1 of lecture notes of week two). Namely, let $\boxed{u=cy+dt+m}$. Then

$$y=\frac{1}{c}(u-dt-m) \Rightarrow y'=\frac{1}{c}(u'-d)$$

$$\Rightarrow \frac{1}{c}(u'-d)=k+\frac{A}{u} \Rightarrow u'=\underbrace{ck+d}_{B}+\frac{\overbrace{cA}^{E}}{u}$$

$$u'=B+\frac{E}{u}$$

1) Case 1: $B=0$ $\quad u'=\frac{E}{u} \overset{\text{Separate}}{\Leftrightarrow} u\,du=E\,dx \overset{\text{Integrate}}{\Leftrightarrow}$

$$\frac{1}{2}u^2=Ex+\text{const} \Rightarrow u=\pm\sqrt{2Ex+\text{const}} \Rightarrow$$

$$y=\frac{1}{c}\left(\pm\sqrt{2Ex+\text{const}}-dt-m\right)$$

Returning to old parameters

$B=0 \Leftrightarrow ck+d=0 \Leftrightarrow a+d=0$ $\quad E=cA=cm-\underbrace{ck}_{a}n=\boxed{cm-an}$

So $\boxed{\begin{array}{l} \text{If } a+d=0 \\ y=\frac{1}{c}\left(\pm\sqrt{2(cm-an)x}-dt-m\right)\end{array}}$

2) Case 2 : $B\neq 0$

$$u'=\frac{Bu+E}{u} \xRightarrow{\text{separate}} \frac{u\,du}{Bu+E}=dx$$

$$\frac{u}{Bu+E}=\frac{\cancel{u+\frac{E}{B}}}{B\cancel{(u+\frac{E}{B})}}-\frac{E}{B}\frac{1}{Bu+E}=\frac{1}{B}-\frac{E}{B}\frac{1}{Bu+E}$$

$$\Rightarrow\int\frac{u\,du}{Bu+E}=\frac{u}{B}-\frac{E}{B^2}\ln|Bu+E| \Rightarrow$$

$$\frac{u}{B}-\frac{E}{B^2}\ln|Bu+E|=x+\text{const}$$

returning $\Rightarrow$ original variables

$$\boxed{\frac{cy+dt+m}{a+d}-\frac{cm-an}{(a+d)^2}\ln|(a+d)(cy+dt+m)+cm-an| = x+\text{const}, \quad \text{if } a+d\neq 0}$$