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Homework #2 Solutions MATH 308 — regular part

Problem 1 (a) $(t^2-1)y' = ty + 2t(t^2-1)$, $|t|>1$

$\Rightarrow \quad y' = \frac{t}{t^2-1}y + 2t \Rightarrow$

$y' - \frac{t}{t^2-1}y = 2t \Rightarrow p(t) = -\frac{t}{t^2-1} \Rightarrow$

The integrating factor μ satisfies $\mu' = -\frac{t}{t^2-1}\mu \Rightarrow$ we can take

$\mu = e^{-\int\frac{t}{t^2-1}dt} = e^{-\frac{1}{2}\ln|t^2-1|} = (\underbrace{e^{\ln(t^2-1)}}_{t^2-1})^{-1/2} = \frac{1}{\sqrt{t^2-1}}$

$\int\frac{t}{t^2-1}dt = \frac{1}{2}\ln|t^2-1|$

Multiply $y' - \frac{t}{t^2-1}y = 2t$ by $\mu = \frac{1}{\sqrt{t^2-1}} \Rightarrow$

$\left(\frac{1}{\sqrt{t^2-1}}y\right)' = \frac{2t}{\sqrt{t^2-1}} \xrightarrow{\text{Integrate}} \frac{1}{\sqrt{t^2-1}}y = \int\frac{2t}{\sqrt{t^2-1}}dt =$

$u = t^2-1$

$du = 2t$

$= \int\frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{t^2-1} + C \Rightarrow \boxed{y(t) = 2(t^2-1) + C\sqrt{t^2-1}}$

(b) Solve the initial value problem

$xy' + 3y = \cos x$, $y(\pi) = \frac{2}{\pi^3}$

$\Rightarrow \quad y' + \frac{3}{x}y = \frac{\cos x}{x} \Rightarrow p(x) = \frac{3}{x} \Rightarrow$ the integrating factor

satisfies $\mu' = \frac{3}{x}\mu \Rightarrow$ (we can take $\mu = e^{\int\frac{3}{x}dx} = e^{3\ln x} = x^3$

Multiplying $y' + \frac{3}{x}y = \frac{\cos x}{x}$ by x^3 we get $(x^3y)' = x^2\cos x \Rightarrow$

$x^3y = \int x^2\cos x\,dx \underset{\text{integration by parts}}{=} x^2\sin x - 2\int x\sin x\,dx \underset{\text{by parts again}}{=} x^2\sin x - 2(-x\cos x +$

$u = x^2,\ v' = \cos x$
$u' = 2x,\ v = \sin x$

$u = x\ \ v' = \sin x$
$u' = 1\ \ v = -\cos x$

② $+\int \cos x\,dx\Big) = x^2\sin x + 2x\cos x - 2\sin x + C \Rightarrow$

The general solution is
$$y(x) = \frac{\sin x}{x} + \frac{2\cos x}{x^2} - \frac{2\sin x}{x^3} + \frac{C}{x^3}$$

$$\frac{2}{\pi^3} = y(\pi) = -\frac{2}{\pi^2} + \frac{C}{\pi^3} \Rightarrow \frac{C}{\pi^3} = \frac{2}{\pi^3} + \frac{2}{\pi^2} \Rightarrow C = 2 + 2\pi = 2(\pi+1)$$

here we use that $\sin\pi = 0$, $\cos\pi = -1$

$\Rightarrow$ the solution of IVP is

$$\boxed{y(x) = \frac{\sin x}{x} + \frac{2\cos x}{x^2} - \frac{2\sin x}{x^3} + \frac{2(\pi+1)}{x^3}}$$

Problem 2 (a) $Q' = r\gamma(t) - \frac{rQ}{V}$, $Q(0) = Q_0$

In our case $V = 200$ gal
$\gamma(t) = 0.2 + 0.4\sin 4t$ oz/gal (here $\gamma(t)$ might be negative which does not have a physical meaning, but it does not matter)
$r = 10$ gal/min
$Q_0 = 60$ oz
$\Rightarrow r\gamma(t) = 2 + 4\sin 4t$, $\frac{r}{V} = \frac{1}{20}$

$\Rightarrow$ the IVP is

$$\boxed{\begin{aligned} Q' &= 2 + 4\sin 4t - \frac{1}{20}Q \\ Q(0) &= 60 \text{ oz}\end{aligned}}$$

(b) $Q' + \frac{1}{20}Q = 2 + 4\sin 4t$. Integrating factor μ satisfies $\mu' = \frac{1}{20}\mu$
$\Rightarrow \mu$ can be taken $\mu = e^{\frac{t}{20}}$ $\Rightarrow$ multiplying the equation by $e^{t/20}$
we get $\left(e^{t/20}Q\right)' = 2e^{t/20} + 4e^{t/20}\sin 4t \Rightarrow$ (*)

$$e^{t/20}Q = \int\left(2e^{t/20} + 4e^{t/20}\sin 4t\right)dt =$$

$\int e^{t/20}\,dt = 20e^{t/20}$

$u = \sin 4t$, $v' = e^{t/20}$
$u' = 4\cos 4t$, $v = 20e^{t/20}$

integration by parts using

$\int e^{t/20}\sin 4t\,dt = 20e^{t/20}\sin 4t - 80\int e^{t/20}\cos 4t\,dt =$

$u = \cos 4t$, $v' = e^{t/20}$
$u' = -4\sin 4t$, $v = 20e^{t/20}$

③ $= 20e^{t/20}\sin 4t - 80\left(20e^{t/20}\cos 4t + 80\int e^{t/20}\sin 4t\,dt\right) =$

$$= 20e^{t/20}\sin 4t - 1600e^{t/20}\cos 4t - 6400\int e^{t/20}\sin 4t \left(= \underbrace{\int e^{t/20}\sin 4t\,dt}_{\text{loop}}\right)$$

$$\Rightarrow 6401\int e^{t/20}\sin 4t = 20e^{t/20}\sin 4t - 1600e^{t/20}\cos 4t \Rightarrow$$

$$\int e^{t/20}\sin 4t = \frac{20}{6401}e^{t/20}\sin 4t - \frac{1600}{6401}e^{t/20}\cos 4t$$

$\Rightarrow$ returning to the formula (*) on the previous page

$$e^{t/20}Q = 40e^{t/20} + \frac{80}{6401}e^{t/20}\sin 4t - \frac{6400}{6401}e^{t/20}\cos 4t + C$$

$$\Rightarrow Q(t) = 40 + \frac{80}{6401}\sin 4t - \frac{6400}{6401}\cos 4t + Ce^{-t/20}$$

$$Q(0) = 60 \Rightarrow \quad 60 = 40 - \frac{6400}{6401} + C \Rightarrow \quad \underbrace{C = 20 + \frac{6400}{6401}}_{\text{not necessary to specify}}$$

$$\Rightarrow \boxed{Q(t) = 40 + \frac{80}{6401}\sin 4t - \frac{6400}{6401}\cos 4t + \left(20 + \frac{6400}{6401}\right)e^{-t/20}}$$

Problem 3 (Problem 4 for Honors section)

$$y(\cos 2x)e^{xy} - 2(\sin 2x)e^{xy} + 2x = \left(3 - x(\cos 2x)e^{xy}\right)\frac{dy}{dx}$$

$$\Leftrightarrow \left(y(\cos 2x)e^{xy} - 2(\sin 2x)e^{xy} + 2x\right)dx - \left(3 - x(\cos 2x)e^{xy}\right)dy = 0$$

$$\Rightarrow P = y(\cos 2x)e^{xy} - 2(\sin 2x)e^{xy} + 2x \Rightarrow P_y = \cos 2x\, e^{xy} + xy(\cos 2x)e^{xy} - 2x(\sin 2x)e^{xy}$$

$$Q = -\left(3 - x(\cos 2x)e^{xy}\right) \Rightarrow Q_x = \cos 2x\, e^{xy} - 2x\sin 2x\, e^{xy} + xy(\cos 2x)e^{xy}$$

So $P_y = Q_x$ at any (x,y) $\Rightarrow$ $\boxed{\text{the equation is exact}}$

④ Find the potential

$$\begin{cases} \varphi_x = y(\cos 2x)e^{xy} - 2(\sin 2x)e^{xy} + 2x & (\text{eq } 1) \\ \varphi_y = -3 + x(\cos 2x)e^{xy} & (\text{eq } 2) \end{cases} \Rightarrow \varphi = -3y + \int x\cos 2x\, e^{xy}\,dy +$$

$$+ h(x) = -3y + \cancel{x}(\cos 2x)\frac{e^{xy}}{\cancel{x}} + h(x) = -3y + \cos 2x\, e^{xy} +$$

$$+ h(x)$$

Substituting into the first equation:

$$\varphi_x = \frac{\partial}{\partial x}\left(-3y + \cos 2x\, e^{xy} + h(x)\right) = -2\sin 2x\, e^{xy} +$$

$$+ \cancel{y\cos 2x\, e^{xy}} + h'(x) = \cancel{y(\cos 2x)e^{xy}} \cancel{- 2(\sin 2x)e^{xy}} + 2x$$

comparing with the R.H.S of (eq 1)

$$\Rightarrow h'(x) = 2x \Rightarrow h(x) = x^2 + C.$$ We can take $C = 0$

So, the potential φ can be taken in the form

$$\varphi = -3y + \cos 2x\, e^{xy} + x^2$$ and the general solution of our ODE is

$$-3y + \cos 2x\, e^{xy} + x^2 = C$$

The initial condition $y(0) = 0$ implies

$$1 = C \Rightarrow C = 1$$ $\Rightarrow$ the solution of IVP is given implicitly by

$$-3y + \cos 2x\, e^{xy} + x^2 = 1$$

(5) Problem 4 (Problem 5 of honors section)

$$\left(x+ye^{2xy}\right)dx+axe^{2xy}dy=0$$

$P = x+ye^{2xy} \Rightarrow P_y = e^{2xy}+2xye^{2xy}$

$Q = axe^{2xy} \Rightarrow Q_x = a\left(e^{2xy}+2xye^{2xy}\right)$

So the condition for exact $P_y = Q_x$ holds $\Leftrightarrow$ $\boxed{a=1}$

Find the general solution for $a=1$:

$$\begin{cases} \varphi_x = x+ye^{2xy} \\ \varphi_y = xe^{2xy} \end{cases} \Rightarrow \varphi = \int\left(x+ye^{2xy}\right)dx + h(y) =$$

$$= \frac{x^2}{2} + \cancel{y}\frac{1}{2\cancel{y}}e^{2xy}+h(y) =$$

$$= \frac{x^2}{2}+\frac{1}{2}e^{2xy}+h(y)$$

Substituting into the second equation:

$$\varphi_y = \frac{\partial}{\partial y}\left(\frac{x^2}{2}+\frac{1}{2}e^{2xy}+h(y)\right) = xe^{2xy}+h'(y) =$$

$$= xe^{2xy} \Rightarrow h'(y)=0 \Rightarrow \text{we can take } h(y)=0$$

$\Rightarrow \quad \varphi = \frac{x^2}{2}+\frac{1}{2}e^{2xy} \Rightarrow$ the general solution is

$$\frac{x^2}{2}+\frac{1}{2}e^{2xy}=C_1 \Leftrightarrow x^2+e^{2xy}=\underbrace{2C_1}_{C} \Leftrightarrow \boxed{x^2+e^{2xy}=C}$$

Actually we can express y as a function of x here:

$$e^{2xy}=C-x^2 \Rightarrow 2xy = \ln(C-x^2) \Rightarrow \boxed{y=\frac{1}{2x}\ln\left(C-x^2\right)}$$

$|x|<\sqrt{|C|}$, $x\neq 0$

6 **Problem 5** (Problem 6(a) for the honors section)

$$x+e^y+\left(\frac{x^2}{2}+2xe^y\right)\frac{dy}{dx}=0$$

$P=x+e^y$ $\qquad P_y=e^y$ $\qquad P_y \neq Q_x \to$ not exact

$Q=\frac{x^2}{2}+2xe^y$ $\qquad Q_x=x+2e^y$

In order that the integrating factor will depend on y only we need to check if the expression $\frac{Q_x-P_y}{P}$ depends on y only: $\frac{Q_x-P_y}{P}=\frac{x+2e^y-e^y}{x+e^y}=\frac{x+e^y}{x+e^y}=\underline{1}$ indeed is independent of x.

$\Rightarrow$ the integrating factor μ can be found such that

$$\frac{d\mu}{dy}=\mu \Rightarrow \mu=e^y$$

Multiply the equation by $\mu=e^y$:

$$xe^y+e^{2y}+\left(\frac{x^2}{2}e^y+2xe^{2y}\right)\frac{dy}{dx}=0$$

Find the potential:

$$\begin{cases}\varphi_x=xe^y+e^{2y}\\ \varphi_y=\frac{x^2}{2}e^y+2xe^{2y}\end{cases} \Rightarrow \varphi=\int(xe^y+e^{2y})dx=\frac{x^2}{2}e^y+xe^{2y}+h(y)$$

$\Rightarrow$ substituting into the second equation:

$$\varphi_y=\frac{\partial}{\partial y}\left(\frac{x^2}{2}e^y+xe^{2y}+h(y)\right)=$$

$$=\cancel{\frac{x^2}{2}e^y}+\cancel{2xe^{2y}}+h'(y)=\cancel{\frac{x^2}{2}e^y}+\cancel{2xe^{2y}}\Rightarrow h'(y)=0$$

$\Rightarrow$ we can take $h(y)=0 \Rightarrow \varphi=\frac{x^2}{2}e^y+xe^{2y} \Rightarrow$ the general solution:

$$\boxed{\frac{x^2}{2}e^y+xe^{2y}=C}$$