1) Homework #10 solutions MATH 308

(a) i) $\begin{cases} x_1' = -5x_1 + 15 \\ x_2' = -5x_2 - 10 \end{cases}$

The critical point: $\begin{cases} -5x_1 + 15 = 0 \Rightarrow x_1 = 3 \\ -5x_2 - 10 = 0 \Rightarrow x_2 = -2 \end{cases}$ $\Rightarrow$ the critical point is $\boxed{(3,-2)}$

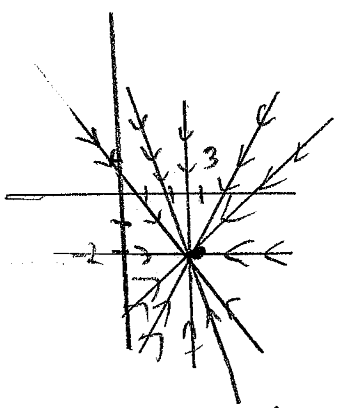
$A = \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$ → the diagonal matrix

Eigenvalues: $\lambda_1 = \lambda_2 = -5$

Geom. multiplicity: $A - (-5I) = 0 \Rightarrow$ geometric multiplicity $= 2$

$\Rightarrow$ $(3,-2)$ is a $\boxed{\text{proper node}}$. Since $\lambda_{1,2} < 0$, it is $\boxed{\text{asymptotically stable}}$

ii)



(b) i) $\begin{cases} x_1' = -9x_1 + 6x_2 + 3 \\ x_2' = 4x_1 + x_2 - 16 \end{cases}$

The critical point: $\begin{cases} -9x_1 + 6x_2 + 3 = 0 \\ 4x_1 + x_2 - 16 = 0 \end{cases} \Leftrightarrow \begin{cases} -3x_1 + 2x_2 = -1 \\ 4x_1 + x_2 = 16 \end{cases}$

$\left(\begin{array}{cc|c} -3 & 2 & -1 \\ 4 & 1 & 16 \end{array}\right) \xrightarrow{R_2 \to 2R_2 - R_1} \left(\begin{array}{cc|c} -3 & 2 & -1 \\ 11 & 0 & 33 \end{array}\right) \Rightarrow \begin{array}{l} -3x_1 + 2x_2 = -1 \quad -9 + 2x_2 = -1 \Rightarrow x_2 = 4 \\ 11x_1 = 33 \Rightarrow x_1 = 3 \end{array}$

$\Rightarrow$ the critical point is $\boxed{(3,4)}$

$A = \begin{pmatrix} -9 & 6 \\ 4 & 1 \end{pmatrix}$, Eigenvalues: $\det(A - \lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A =$

$= \lambda^2 + 8\lambda - 33 = 0$

$D = 64 + 132 = 196 = 14^2 \Rightarrow \lambda_1 = \frac{-8+14}{2} = 3$

$\lambda_2 = \frac{-8-14}{2} = -11$

② The eigenvalues are real and of opposite signs $\Rightarrow$ $\boxed{(3,4)}$ is a saddle $\Rightarrow$ $\boxed{\text{unstable}}$

ii) To sketch the phase portrait find the eigenlines:

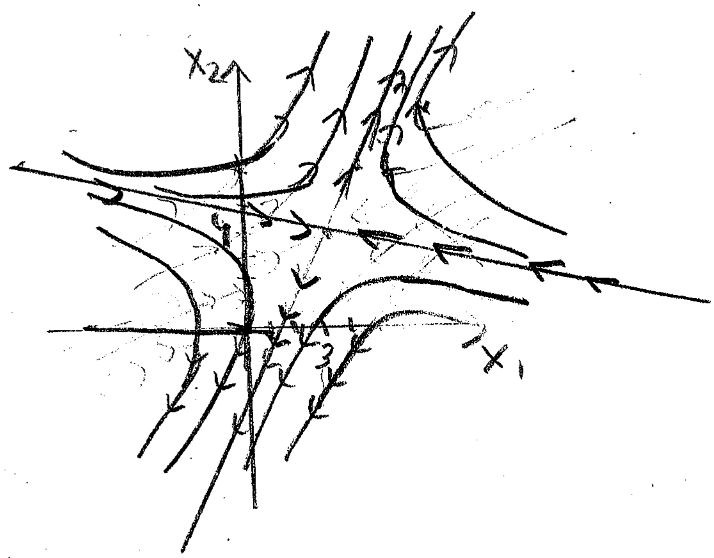
• $\lambda = 3$: $(A - 3I)v = \begin{pmatrix} -12 & 6 \\ 4 & -2 \end{pmatrix} v = 0 \Leftrightarrow 2v_1 - v_2 = 0 \Rightarrow v_2 = 2v_1$

the eigenline $E_3 = \{ c \begin{pmatrix} 1 \\ 2 \end{pmatrix}, c \in \mathbb{R} \}$ $\rightarrow$ unstable separatrix

• $\lambda = -11$ $(A - (-11)I) = \begin{pmatrix} 2 & 6 \\ 4 & 12 \end{pmatrix} v = 0 \Leftrightarrow v_1 + 3v_2 = 0 \Rightarrow v_1 = 3v_3 \Rightarrow$

the eigenline $E_{-11} = \{ c \begin{pmatrix} 3 \\ -1 \end{pmatrix}, c \in \mathbb{R} \}$ $\rightarrow$ stable separatrix

Sketch



(c) (i) $\begin{cases} x_1' = 3x_1 - 4x_2 \\ x_2' = 5x_1 + 7x_2 - 41 \end{cases}$

The critical point: $\begin{cases} 3x_1 - 4x_2 = 0 \\ 5x_1 + 7x_2 - 41 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x_1 - 4x_2 = 0 \\ 5x_1 + 7x_2 = 41 \end{cases}$

$\left(\begin{array}{cc|c} 3 & -4 & 0 \\ 5 & 7 & 41 \end{array} \right) \xrightarrow{R_2 = 3R_2 - 5R_1} \left(\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 41 & 3 \cdot 41 \end{array} \right) \Rightarrow$ $3x_1 - 4x_2 = 0$, $x_2 = 3$ $\Rightarrow$ $x_1 = 4$ $\Rightarrow$ $\boxed{(4,3)}$ is the critical point

$A = \begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix}$. Eigenvalues: $\det(A - \lambda I) = \lambda^2 - 10\lambda + 41 = 0$

$\Delta = 100 - 164 = -64$

$\lambda_{1,2} = \frac{10 \pm 8i}{2} = 5 \pm 4i \rightarrow$

Complex eigenvalues with positive real parts $\Rightarrow (4,3)$ is

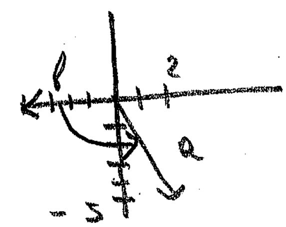
a $\boxed{\text{spiral source}}$ $\Rightarrow$ $\boxed{\text{unstable}}$

(ii) To understand the direction of motion along the spiral we need to use some information about the eigenvectors

Find an eigenvector: $\left(A-(5+4i)I\right)v = \begin{pmatrix} -2-4i & -4 \\ 5 & 2-4i \end{pmatrix} v \Leftrightarrow$

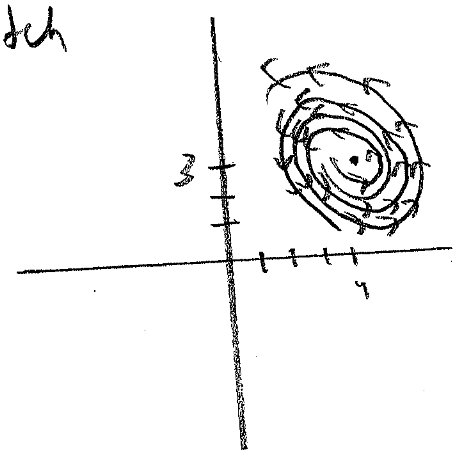
$\Leftrightarrow 5v_1 + (2-4i)v_2 = 0 \Rightarrow$ we can take $v_1 = (2-4i)$, $v_2 = -5$

$\Rightarrow v = \begin{pmatrix} 2-4i \\ -5 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ -5 \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} -4 \\ 0 \end{pmatrix}}_{b} i$



$\Rightarrow$ the shortest way from b to a is counterclockwise

The sketch



(d) $\begin{cases} x_1' = 6x_1 - 13x_2 + 1 \\ x_2' = x_1 - 8x_2 + 6 \end{cases}$

Critical points: $\begin{cases} 6x_1 - 13x_2 + 1 = 0 \\ x_1 - 8x_2 + 6 = 0 \end{cases}$ $\begin{cases} 6x_1 - 13x_2 = -1 & (\text{Eq } 1) \\ x_1 - 8x_2 = -6 & (\text{Eq } 2) \end{cases}$

$(\text{Eq } 1) - 6(\text{Eq } 2)$: $\underbrace{(-13+48)}_{35} x_2 = \underbrace{-1+36}_{35} \Rightarrow x_2 = 1 \Rightarrow 6x_1 - 13 = -1 \Rightarrow$

$6x_1 = 12 \Rightarrow x_1 = 2 \Rightarrow$ the critical point is $\boxed{(2,1)}$

$A = \begin{pmatrix} 6 & -13 \\ 1 & -8 \end{pmatrix}$ Eigenvalues: $\det(A - \lambda I) = \lambda^2 + 2\lambda - 35 = 0$

$D = 4 + 140 = 144$

$\lambda_1 = \frac{-2+12}{2} = 5$

$\lambda_2 = \frac{-2-12}{2} = -7$

The eigenvalues are real and of opposite signs $\Rightarrow$ $\boxed{\text{saddle point}}$ $\rightarrow$ $\boxed{\text{unstable}}$

ii) To sketch the phase portrait find the eigenlines:

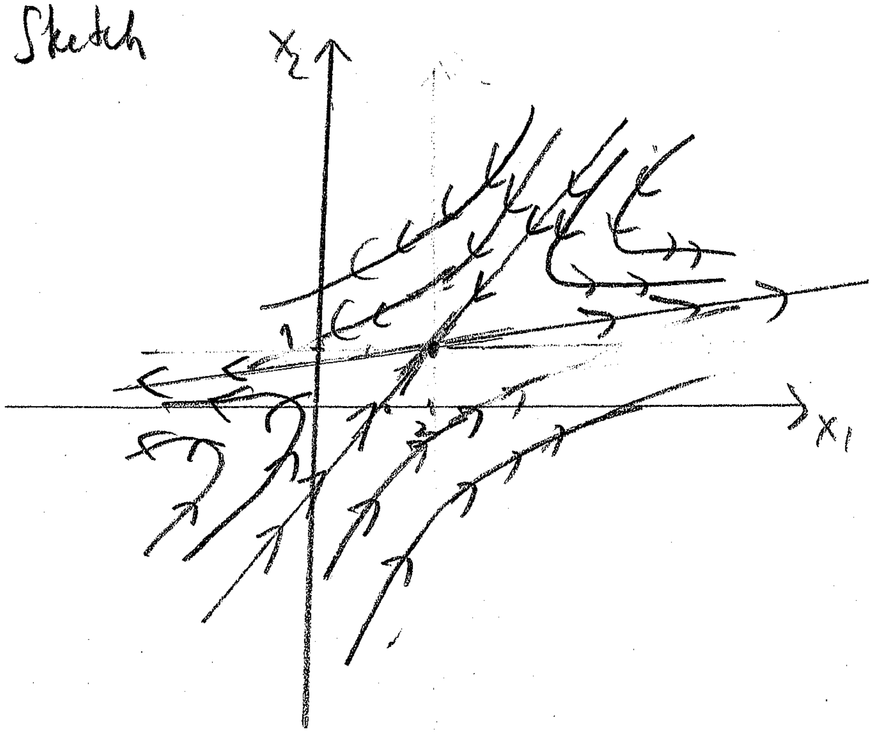
- $\lambda = 5$ $(A - 5I)v = \begin{pmatrix} 1 & -13 \\ 1 & -13 \end{pmatrix} v = 0 \Leftrightarrow v_1 - 13v_2 = 0 \Leftrightarrow v_1 = 13v_2$

$\Rightarrow E_5 = \{c \begin{pmatrix} 13 \\ 1 \end{pmatrix} : c \in \mathbb{R}\}$ $\rightarrow$ unstable separatrix

- $\lambda = -7$ $(A - (-7)I)v = \begin{pmatrix} 13 & -13 \\ 1 & -1 \end{pmatrix} v = 0 \Leftrightarrow v_1 - v_2 = 0 \Leftrightarrow v_1 = v_2$

$\Rightarrow E_{-7} = \{c \begin{pmatrix} 1 \\ 1 \end{pmatrix} : c \in \mathbb{R}\}$ $\rightarrow$ stable separatrix

Sketch



(e) (i) $x_1' = -10x_1 + 6x_2 + 2$

$x_2' = -6x_1 + 2x_2 + 14$

Critical points $\begin{cases} -10x_1 + 6x_2 = -2 \\ -6x_1 + 2x_2 = -14 \end{cases} \Leftrightarrow \begin{cases} -5x_1 + 3x_2 = -1 & (Eq\ 1) \\ -3x_1 + x_2 = -7 & (Eq\ 2) \end{cases}$

$(Eq\ 1) - 3(Eq\ 2): (-5+9)x_1 = -1 + 21 \Leftrightarrow 4x_1 = 20 \Leftrightarrow x_1 = 5$

$\Rightarrow -25 + 3x_2 = -1 \Rightarrow 3x_2 = 24 \Rightarrow x_2 = 8$

$\Rightarrow$ $\boxed{(5,8) \text{ is a critical point}}$

$A = \begin{pmatrix} -10 & 6 \\ -6 & 2 \end{pmatrix}$ Eigenvalues: $\lambda^2 + 8\lambda + 16 = 0 \Leftrightarrow$

$(\lambda+4)^2 = 0 \Rightarrow \lambda_{1,2} = -4$

$A \neq -4I$ so geometric multiplicity of $\lambda = -4$ is 1 $\Rightarrow$

improper node. Since $\lambda_{1,2} < 0$, it is $\boxed{\text{improper nodal sink}}$

$\Rightarrow$ $\boxed{\text{asymptotically stable}}$

(ii) For sketch we need to find an eigenline and we need some information about a pair (v, w) where v is an eigenvector and $(A - \lambda I)w = v$

Eigenvectors: $(A - (-4I))v = (A + 4I)v = \begin{pmatrix} -6 & 6 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow$

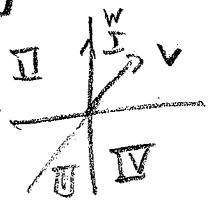
$\Leftrightarrow -v_1 + v_2 = 0 \Leftrightarrow v_2 = v_1 \Leftrightarrow$ $E_{-4} = \left\{ c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, c \in \mathbb{R} \right\}$

Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A + 4I)w = v \Leftrightarrow \begin{pmatrix} -6 & 6 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow$ $-6w_1 + 6w_2 = 1$

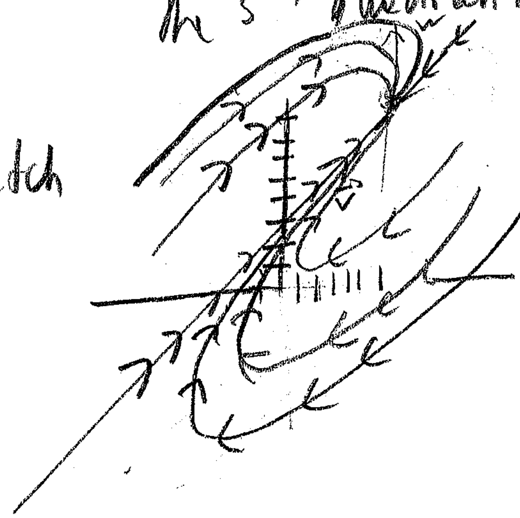
Let $w_1 = 0 \Rightarrow w_2 = \frac{1}{6}$

$w = \begin{pmatrix} 0 \\ 1/6 \end{pmatrix}$



According to the rule: since $\lambda < 0$ the trajectories entering the origin from the 1st and the 3rd quadrant with respect to the pair (v, w)

Sketch



Problem 2

$$x' = x(16-3x-2y)$$
$$y' = y(26-4y-5x)$$

(a) We have to solve the system

$$\begin{cases} x(16-3x-2y)=0 & \Rightarrow x=0 \text{ or } 16-3x-2y=0 \\ y(26-4y-5x)=0 & \Rightarrow y=0 \quad \text{or } 26-4y-5x=0 \end{cases}$$

We have 4 options

i) $x=0, y=0 \Rightarrow (0,0)$ is a critical point

ii) $\begin{cases} x=0 \\ 26-4y-5x=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ 26-4y=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=\frac{13}{2} \end{cases} \Rightarrow \left(0, \frac{13}{2}\right)$ is a critical point

ii) $\begin{cases} 16-3x-2y=0 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} 16-3x=0 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{16}{3} \\ y=0 \end{cases} \Rightarrow \left(\frac{16}{3}, 0\right)$ is a critical point

7) iv) $\begin{cases} 16-3x-2y=0 \\ 26-4y-5x=0 \end{cases} \Leftrightarrow \begin{cases} 3x+2y=16 & (Eq\ 1) \\ 5x+4y=26 & (Eq\ 2) \end{cases}$

(Eq 2) − 2(Eq 1): $5x-6x=26-32 \Rightarrow -x=-6 \Rightarrow x=6 \Rightarrow$

$18+2y=16 \Leftrightarrow 2y=-2 \Rightarrow y=-1 \Rightarrow (6,-1)$ is a critical point $\Rightarrow$ Critical points are $\boxed{(0,0),\left(0,\frac{13}{2}\right),\left(\frac{16}{3},0\right),(6,-1)}$

8) Let $f(x,y)=16x-3x^2-2xy$
$g(x,y)=26y-4y^2-5xy$

The Jacobi matrix $J(x,y)=\begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix}=$

$=\begin{pmatrix} 16-6x-2y & -2x \\ -5y & 26-8y-5x \end{pmatrix}$

i) For $(0,0)$
$J(0,0)=\begin{pmatrix} 16 & 0 \\ 0 & 26 \end{pmatrix} \Rightarrow$ the corresponding linear system near $(0,0)$ is $\begin{cases} u_1'=16u_1 \\ u_2'=26u_2 \end{cases}$

ii) For $\left(0,\frac{13}{2}\right)$
$J\left(0,\frac{13}{2}\right)=\begin{pmatrix} 16-13 & 0 \\ -\frac{65}{2} & 26-52 \end{pmatrix}=\begin{pmatrix} 3 & 0 \\ -\frac{65}{2} & -26 \end{pmatrix} \Rightarrow$

the corresponding linear system near $\left(0,\frac{13}{2}\right)$ is $\begin{cases} u_1'=3u_2 \\ u_2'=-\frac{65}{2}u_1-26u_2 \end{cases}$

iii) For $\left(\frac{16}{3}, 0\right)$

$$J\left(\frac{16}{3}, 0\right) = \begin{pmatrix} 16 - \overbrace{6 \cdot \frac{16}{3}}^{32} & -\frac{32}{3} \\ 0 & 26 - \underbrace{5 \cdot \frac{16}{3}}_{\frac{80}{3}} \end{pmatrix} =$$

$$= \begin{pmatrix} -16 & -\frac{32}{3} \\ 0 & -\frac{2}{3} \end{pmatrix}$$ $\Rightarrow$ the corresponding linear system near $\left(\frac{16}{3}, 0\right)$ is

$$\begin{cases} u_1' = -16u_1 - \frac{32}{3}u_2 \\ u_2' = -\frac{2}{3}u_2 \end{cases}$$

(iv) For $(6, -1)$

$$J(6, -1) = \begin{pmatrix} 16 - 36 + 2 & -12 \\ 5 & 26 + 8 - 30 \end{pmatrix} = \begin{pmatrix} -18 & -12 \\ 5 & 4 \end{pmatrix}$$

$\Rightarrow$ the corresponding linear system near $(6, -1)$ is

$$\begin{cases} u_1' = -18u_1 - 12u_2 \\ u_2' = 5u_1 + 4u_2 \end{cases}$$

c) i) Case of $(0,0)$: $J(0,0) = \begin{pmatrix} 16 & 0 \\ 0 & 26 \end{pmatrix}$ $\rightarrow$ the diagonal matrix

The eigenvalues are $\lambda_1 = 16$, $\lambda_2 = 26$ $\rightarrow$ real, distinct and both positive $\Rightarrow$ $\boxed{\text{nodal source}}$ $\boxed{\text{unstable}}$

ii) Case of $\left(0, \frac{13}{2}\right)$

$$J\left(0, \frac{13}{2}\right) = \begin{pmatrix} 3 & 0 \\ -\frac{65}{2} & -26 \end{pmatrix} \to \text{lower triangular matrix}$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -26$ → real and of opposite sign ⇒ $\boxed{\text{saddle point}}$, $\boxed{\text{unstable}}$

iii) Case of $\left(\frac{16}{3}, 0\right)$

$$J\left(\frac{16}{3}, 0\right) = \begin{pmatrix} -16 & -\frac{32}{3} \\ 0 & -\frac{3}{2} \end{pmatrix} \to \text{upper triangular matrix}$$

The eigenvalues are $\lambda_1 = -16$, $\lambda_2 = -\frac{3}{2}$ ⇒ real, distinct, negative ⇒ $\boxed{\text{nodal sink}}$, $\boxed{\text{asymptotically stable}}$

iv) Case of $(6, -1)$

$$J(6,-1) = \begin{pmatrix} -18 & -12 \\ 5 & 4 \end{pmatrix}$$

Eigenvalues: $\lambda^2 + 14\lambda - 12 = 0$

$\operatorname{tr} J(6,-1) = -14$

$\det J(6,-1) = -72 + 60 = -12$

$D/4 = 49 + 12 = 61 > 0$

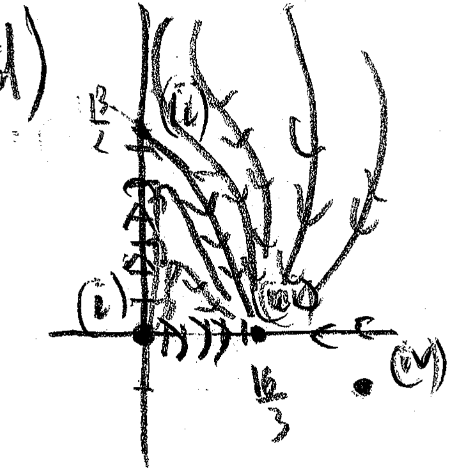
$\lambda_1 = -7 + \sqrt{61} > 0$

$\lambda_2 = -7 - \sqrt{61} < 0$ ⇒

The eigenvalues are real and of opposite sign ⇒

$\boxed{\text{saddle point}}$, unstable

d)



It is useful to find the

eigenvectors of saddle point $(0, \frac{13}{2})$

corresp. to unstable separatrix

$\lambda = 3$: $\left(J(0, \frac{13}{2}) - 3I\right) = \begin{pmatrix} 0 & 0 \\ -\frac{65}{2} & -29 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\frac{65}{2} v_1 + 29 v_2 = \Rightarrow$ if $v_1 = 29$, $v_2 = -\frac{65}{2} = -32.5$

e) This item **is cancelled**, because originally I meant to give a problem such that the forth critical point will have both components being positive but it is not the case.
This immediately implies that there is no coexistence and from the analysis it follows that the x-species will win and the y-species will be extincted.