Homework #11 Solutions MATH 308-505

Problem 1

$$f(t) = 1 + (t+1-1)\,u_3(t) + (1-3t-t-1)\,u_5(t) =$$

$$= 1 + t\,u_3(t) - 4t\,u_5(t)$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\,u_3(t)\} = \mathcal{L}\{(t-3+3)u_3(t)\} = \mathcal{L}\{(t-3)u_3(t)\} + 3\mathcal{L}\{u_3(t)\} =$$

$$= \frac{e^{-3s}}{s^2} + 3\frac{e^{-3s}}{s} = \frac{3s+1}{s^2}e^{-3s}.$$

(you can also proceed here as follows for finding $\mathcal{L}\{t\,u_3(t)\}$:

Find $f(t)$ s.t. $f(t-3) = t \Rightarrow f(t) = f((t+3)-3) = t+3 \Rightarrow$

$F(s) = \frac{1}{s^2} + \frac{3}{s} \Rightarrow \mathcal{L}\{t\,u_3(t)\} = \left(\frac{1}{s^2} + \frac{3}{s}\right)e^{-3s} = \frac{3s+1}{s^2}e^{-3s}$);

$$\mathcal{L}\{t\,u_5(t)\} = \mathcal{L}\{(t-5)u_5(t)\} + 5\mathcal{L}\{u_5(t)\} = \frac{1}{s^2} + \frac{5}{s} = \frac{5s+1}{s^2}$$

$\Downarrow$

The answer is $\boxed{\frac{1}{s} + \frac{3s+1}{s^2}e^{-3s} - \frac{5s+1}{s^2}e^{-5s}}$

Problem 2

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}(2s+1)}{s^3-2s^2+10s}\right\}$$

First find $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^3-2s^2+10s}\right\}$

Partial fraction decomposition

$$\frac{2s+1}{s^3-2s^2+10s} = \frac{2s+1}{s(s^2-2s+10)} = \frac{2s+1}{s((s-1)^2+9)} = \frac{A}{s} + \frac{B(s-1)+3C}{(s-1)^2+9}$$

$$2s+1 = A((s-1)^2+9) + (B(s-1)+3C)s$$

To find A put $s=0$: $1 = A(1+9) \Rightarrow A = \frac{1}{10}$

To find C put $s = 1$

$$3 = 9A + 3C = \frac{9}{10} + 3C \Rightarrow$$

$$1 = \frac{3}{10} + C \Rightarrow C = \frac{7}{10}$$

To find B comare coefficients of s^2:

$$0 = A + B \Rightarrow B = -A = -\frac{1}{10}$$

$$\Downarrow$$

$$\frac{2s+1}{s^3-2s^2+10s} = \frac{1}{10s} - \frac{1}{10}\frac{s-1}{(s-1)^2+9} + \frac{7}{10}\frac{3}{(s-1)^2+9}$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^3-2s^2+10s}\right\} = \boxed{\frac{1}{10} - \frac{1}{10}e^{t}\cos 3t + \frac{7}{10}e^{t}\sin 3t}$$

<u>Problem 3</u>

$$y'' + 9y = g(t);\ y(0) = -2,\ y'(0) = 1$$

$$g(t) = \sin t + (1 - 2\sin t)\,u_{\frac{\pi}{2}}(t)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\frac{\pi}{2}}(t)\} - 2\mathcal{L}\{u_{\frac{\pi}{2}}(t)\underbrace{\sin t}_{\cos(t-\frac{\pi}{2})}\} =$$

$$= \frac{1}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s} - 2e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + 2s - 1$$

$$\mathcal{L}\{y'' + 9y\} = (s^2+9)Y + 2s - 1 = \frac{1}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s} - \frac{2e^{-\frac{\pi}{2}s}s}{s^2+1} \Rightarrow$$

$$Y = \frac{1}{(s^2+1)(s^2+9)} - \frac{2s-1}{s^2+9} + \frac{1}{s^2+9}\underbrace{\left(\frac{1}{s} - \frac{2s}{s^2+1}\right)}_{\frac{s^2+1-2s^2}{s(s^2+1)} = \frac{1-s^2}{(s^2+1)s}}e^{-\frac{\pi}{2}s}$$

1) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{(u+1)(u+9)} = \frac{1}{8}\left(\frac{1}{u+1} - \frac{1}{u+9}\right) = \frac{1}{8}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

Substitute $u = s^2$

$$\Downarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{8}\cdot\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} =$$

$$= \frac{1}{8}\sin t - \frac{1}{24}\sin 3t$$

2) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+9}\right\}$

$$\frac{2s-1}{s^2+9} = \frac{As+3B}{s^2+9} \Rightarrow \quad A=2,\ B=-\frac{1}{3}$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+9}\right\} = 2\cos 3t - \frac{1}{3}\sin 3t$$

3) $\mathcal{L}^{-1}\left\{\frac{1-s^2}{s(s^2+9)(s^2+1)}\right\}$

$$\frac{1-s^2}{s(s^2+9)(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+3E}{s^2+9}$$

$$1-s^2 = A(s^2+1)(s^2+9) + (Bs+C)s(s^2+9) + (Ds+E)s(s^2+1)$$

To find A put $s=0$: $1 = 9A \Rightarrow \boxed{A = \frac{1}{9}}$

Coefficient of s^4: $0 = A+B+D \Rightarrow B+D = -\frac{1}{9}$

Coefficient of s^3: $0 = C+E$

Coefficient of s^2: $-1 = 10A + 9B + D \Rightarrow -1 = \frac{10}{9} + 9B + D \Rightarrow$

$$9B + D = -\frac{19}{9}$$

$\begin{cases} B+D = -\frac{1}{9} \\ 9B+D = -\frac{19}{9} \end{cases}$ Eq 2 − Eq 1 $\Rightarrow 8B = -\frac{18}{9} = -2 \Rightarrow \boxed{B = -\frac{1}{4}} \Rightarrow D = -\frac{1}{9} + \frac{1}{4} = \boxed{\frac{5}{36}}$

Coefficient of s: $0 = 9C + E \Rightarrow \begin{cases} C+E=0 \\ 9C+E=0 \end{cases} \Rightarrow \boxed{C = E = 0}$

$$\frac{1-s^2}{s(s^2+9)(s^2+1)} = \frac{1}{9}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+1} + \frac{5}{36}\frac{s}{s^2+9} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1-s^2}{s(s^2+9)(s^2+1)}\right\} = \frac{1}{9} - \frac{1}{4}\cos t + \frac{5}{36}\cos 3t$$

$\Downarrow$

4) $$\mathcal{L}^{-1}\left\{\frac{1-s^2}{s(s^2+9)(s^2+1)}e^{-\frac{\pi}{2}s}\right\} = \left(\frac{1}{9} - \frac{1}{4}\underbrace{\cos\left(t-\frac{\pi}{2}\right)}_{\sin t} + \frac{5}{36}\underbrace{\cos 3\left(t-\frac{\pi}{2}\right)}_{\cos\left(3t-\frac{3\pi}{2}\right) = -\sin 3t}\right)u_{\frac{\pi}{2}}(t)$$

$$= \left(\frac{1}{9} - \frac{1}{4}\sin t - \frac{5}{36}\sin 3t\right)u_{\frac{\pi}{2}}(t)$$

$\Downarrow$

Combining 1) - 4):

$$y(t) = \frac{1}{8}\sin t - \frac{1}{24}\sin 3t - 2\cos 3t + \frac{1}{3}\sin 3t -$$

$$-\left(\frac{1}{9} - \frac{1}{4}\sin t - \frac{5}{36}\sin 3t\right)u_{\frac{\pi}{2}}(t) =$$

$$= \boxed{\frac{1}{8}\sin t - \frac{7}{24}\sin 3t - 2\cos 3t - \left(\frac{1}{9} - \frac{1}{4}\sin t - \frac{5}{36}\sin 3t\right)u_{\frac{\pi}{2}}(t)}$$