Homework assignment # 10 solutions, MATH 308

Problem 1

$$\begin{cases} x' = \dfrac{xy-1}{2} \\ y' = 4x - \dfrac{1}{4}y^3 \end{cases}$$

(a) Critical points satisfy the system of equations

$$\begin{cases} \dfrac{xy-1}{2} = 0 & (\text{Eq}1) \\ 4x - \dfrac{1}{4}y^3 = 0 & (\text{Eq}2) \end{cases} \Rightarrow xy = 1 \Rightarrow x = \frac{1}{y} \xrightarrow{(\text{Eq}2)}$$

$$\frac{4}{y} - \frac{1}{4}y^3 = 0 \Leftrightarrow 16 - y^4 = 0 \Leftrightarrow$$

$y^4 = 16 \Rightarrow$ either $y = 2 \Rightarrow x = \frac{1}{2}$, or $y = -2 \Rightarrow x = -\frac{1}{2}$, So we have 2 critical points $\boxed{\left(\frac{1}{2}, 2\right) \text{ and } \left(-\frac{1}{2}, -2\right)}$

(b) First find the Jacobi matrix $J(x,y)$:

$$f(x,y) = \frac{xy-1}{2}, \quad g(x,y) = 4x - \frac{1}{4}y^3 \Rightarrow J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{y}{2} & \frac{x}{2} \\ 4 & -\frac{3}{4}y^2 \end{pmatrix}$$

i) Linear system at $\left(\frac{1}{2}, 2\right)$

$$J\left(\frac{1}{2}, 2\right) = \begin{pmatrix} 1 & \frac{1}{4} \\ 4 & -\frac{3}{4}\cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{4} \\ 4 & -3 \end{pmatrix} \Rightarrow$$

the linear system near $\left(\frac{1}{2}, 2\right)$ is

$$\begin{cases} u' = u + \frac{1}{4}v \\ v' = 4u - 3v \end{cases}$$

ii) Linear system at $\left(-\frac{1}{2}, -2\right)$

$$J\left(-\frac{1}{2}, -2\right) = \begin{pmatrix} -1 & -\frac{1}{4} \\ 4 & -3 \end{pmatrix} \Rightarrow$$

the linear system near $\left(-\frac{1}{2}, -2\right)$ is

$$\begin{cases} u' = -u - \frac{1}{4}v \\ v' = 4u - 3v \end{cases}$$

(c) i) The case of $(\frac{1}{2}, 2)$

$\det J(\frac{1}{2}, 2) = -3 - 4 \cdot \frac{1}{4} = -4 < 0 \Rightarrow (2, \frac{1}{2})$ is

a $\boxed{\text{saddle point}}$ (see Remark on page 15

of class notes of April 25) $\Rightarrow \boxed{\text{unstable}}$

ii) The case of $(-\frac{1}{2}, -2)$

$$\left.\begin{array}{l} \det J(-\frac{1}{2}, -2) = 3 + 1 = 4 \\ \operatorname{tr} J(-\frac{1}{2}, -2) = -4 \end{array}\right\} \Rightarrow \text{the characteristic}$$

equation is $\lambda^2 + 4\lambda + 4 = 0$

$D = 16 - 4 \cdot 4 = 0$!

$\lambda_{1,2} = -\frac{4}{2} = -2 \rightarrow$ repeated eigen value is

Since $J(-\frac{1}{2}, -2) \neq -2I \rightarrow$ $\boxed{\text{improper node for the linearization}}$ $\Rightarrow \boxed{\text{asymptotically stable}}$

<u>Problem 2</u> $\begin{cases} x' = x(22 - 5x - 7y) \\ y' = y(9 - 3y - 2x) \end{cases}$

(a) Critical points satisfy the following system of equations

$\begin{cases} x(22 - 5x - 7y) = 0 \Rightarrow \text{either } x = 0 \text{ or } 5x + 7y = 22 \\ y(9 - 3y - 2x) = 0 \Rightarrow \text{either } y = 0 \text{ or } 2x + 3y = 9 \end{cases}$

Page 3) Four cases

1) $x=0$ & $y=0$ $\Rightarrow$ $(0,0)$ is a critical point

2) $x=0$ & $2x+3y=9$ $\Rightarrow$ $3y=9 \Rightarrow y=3 \Rightarrow$ $(0,3)$ is a critical point

3) $5x+7y=22$ & $y=0$ $\Rightarrow$ $5x=22 \Rightarrow x=\frac{22}{5}=4.4 \Rightarrow (4.4,0)$ is a critical point

4) $\begin{cases} 5x+7y=22 & (Eq\,1) \\ 2x+3y=9 & (Eq\,2) \end{cases}$ $\quad 2(Eq\,1)-5(Eq\,2) \Rightarrow$

$(14-15)y = 44-45 \Rightarrow y=1$

$\overset{Eq\,1}{\Rightarrow}$ $5x+7=22 \Rightarrow x=3 \Rightarrow$

$(3,1)$ is a critical point

So we have 4 critical points $\boxed{(0,0),\ (0,3),\ (4.4,0),\text{ and } (3,1)}$

(B) First find the Jacobi matrix

$f(x,y) = x(22-5x-7y) = 22x-5x^2-7xy$

$g(x,y) = y(9-3y-2x) = 9y-3y^2-2xy$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 22-10x-7y & -7x \\ -2y & 9-6y-2x \end{pmatrix}$$

i) $(x_0,y_0)=(0,0) \Rightarrow J(0,0) = \begin{pmatrix} 22 & 0 \\ 0 & 9 \end{pmatrix} \Rightarrow$ Linear system near $(0,0)$ is $u'=22u$, $v'=9v$

ii) $(x_0,y_0)=(0,3) \Rightarrow J(0,3) = \begin{pmatrix} 22-21 & 0 \\ -6 & 9-18 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -6 & -9 \end{pmatrix} \Rightarrow$

$\Rightarrow$ Linear system near $(0,3)$ is $\begin{cases} u'=u \\ v'=-6u-9v \end{cases}$

iii) $(x_0,y_0) = (4.4,0) \Rightarrow J(4.4,0) = \begin{pmatrix} 22-44 & -7\cdot 4.4 \\ 0 & 9-8.8 \end{pmatrix} =$

$= \begin{pmatrix} -22 & -30.8 \\ 0 & 0.2 \end{pmatrix} \Rightarrow$ Linear system near $(4.4,0)$ is

$\begin{cases} u' = -22u - 30.8v \\ v' = 0.2v \end{cases}$

iv) $(x_0,y_0) = (3,1) \Rightarrow J(3,1) = \begin{pmatrix} 22-30-7 & -21 \\ -2 & 9-6-6 \end{pmatrix} =$

$= \begin{pmatrix} -15 & -21 \\ -2 & -3 \end{pmatrix} \Rightarrow$ Linear system near $(3,1)$ is

$\begin{cases} u' = -15u - 21v \\ v' = -2u - 3v \end{cases}$

Rem Everywhere above $u = x - x_0$, $v = y - y_0$

(c) (i) $(x_0,y_0) = (0,0)$ $\quad J(0,0) = \begin{pmatrix} 22 & 0 \\ 0 & 9 \end{pmatrix}$ is diagonal matrix $\Rightarrow$ the eigenvalues are 22 & 9 $\Rightarrow$ both positive real $\Rightarrow$ $\boxed{\text{nodal source}}$ $\boxed{\text{unstable}}$

(ii) $(x_0,y_0) = (0,3)$ $\quad J(0,3) = \begin{pmatrix} 1 & 0 \\ -6 & -9 \end{pmatrix}$ is a lower triangular matrix $\Rightarrow$ the eigenvalues are 1 & -9 $\Rightarrow$ real and of opposite signs $\Rightarrow$ $\boxed{\text{saddle point}}$ $\boxed{\text{unstable}}$

iii) $(x_0,y_0) = (4.4,0) \Rightarrow J(4.4,0) = \begin{pmatrix} -22 & -30.8 \\ 0 & 0.2 \end{pmatrix}$ is an upper triangular matrix $\Rightarrow$ the eigenvalues are -22 & 0.2 $\Rightarrow$ real and of opposite signs $\Rightarrow$ saddle point unstable

iv) $(x_0,y_0) = (3,1)$. Find the eigenvalues of $J(3,1) = \begin{pmatrix} -15 & -21 \\ -2 & -3 \end{pmatrix}$

$tr\, J = -18$ $\Rightarrow$ characteristic equation is

$\det J = 45 - 42 = 3$ $\quad \lambda^2 + 18\lambda + 3 = 0$

$D = 324 - 12 = 312 > 0 \Rightarrow$

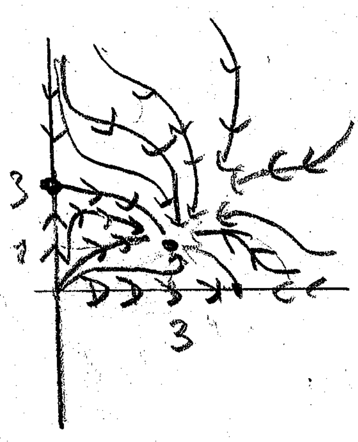
The eigenvalues are <u>real</u>

Also they are <u>negative</u>, because by Vieta theorem $\lambda_1 \lambda_2 = 3 \Rightarrow \lambda_1 \& \lambda_2$ are of the same sign

$\lambda_1 + \lambda_2 = -18 \Rightarrow$ they are both negative

So $\lambda_1 \& \lambda_2$ are negative real $\Rightarrow$ $\boxed{\text{nodal sink}}$.
$\lambda_1 \neq \lambda_2$ $\quad \boxed{\text{asymptotically stable}}$

1d) Note that x-axis is the stable separatrix for the saddle point $(4.4, 0)$
This follows from the fact that $(x(t), 0)$ with $x(t)$ satisfying the one-dimensional population dynamics equation $x' = x(22-5x)$ (as in section 2.5) is the solution of our system
In the same way y-axis is the stable for the saddle point $(0,3)$



d) Since all trajectories starting at a point in the interior of the first quadrant approach the point $(1,3)$ inside the first quadrant the coexistence occurs in this model

(a) $\begin{cases} x_1' = 5x_1 + 4x_2 + (t+2)e^{-2t} \\ x_2' = -2x_1 - 4x_2 - 3e^{-2t} \end{cases}$ (1) $\Rightarrow f(t) = \begin{pmatrix} (t+2)e^{-2t} \\ -3e^{-2t} \end{pmatrix}$

1) Find a fundamental set of solutions for the corresponding homogeneous system

$$\begin{cases} x_1' = 5x_1 + 4x_2 \\ x_2' = -2x_1 - 4x_2 \end{cases} \quad (2)$$

$A = \begin{pmatrix} 5 & 4 \\ -2 & -4 \end{pmatrix}$, $\operatorname{tr} A = 1$, $\det A = -20 + 8 = -12$

. The characteristic equation is $\lambda^2 - \lambda - 12 = 0 \Rightarrow D = 1 + 48 = 49$

$\lambda_1 = \frac{1-7}{2} = -3$

$\lambda_2 = \frac{1+7}{2} = 4$

. An eigenvector of $\lambda = -3$:

$(A + 3I)v = \begin{pmatrix} 8 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow 2v_1 + v_2 = 0 \Rightarrow v^1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector of $\lambda_1 = -3 \Rightarrow$

$\Rightarrow x^1(t) = e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix}$ is a solution of (2)

. An eigenvector of $\lambda = 4$

$(A - 4I)v = \begin{pmatrix} 1 & 4 \\ -2 & -8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow v_1 + 4v_2 = 0 \Rightarrow v^2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is an eigenvector of $\lambda_2 = 4 \Rightarrow$

$x^2(t) = e^{4t}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4e^{4t} \\ -e^{4t} \end{pmatrix}$ is a solution of (2)

So $\begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix}$ & $\begin{pmatrix} 4e^{4t} \\ -e^{4t} \end{pmatrix}$ for a fundamental set of solutions of (2) $\Rightarrow$ $\Psi(t) = \begin{pmatrix} e^{-3t} & 4e^{4t} \\ -2e^{-3t} & -e^{4t} \end{pmatrix}$ is a fundamental matrix for (2)

2) Find a solution of (1) in the form

$$x(t) = \Psi(t)\, u(t) \quad \text{where } u'(t) \text{ satisfies}$$

$$\Psi(t)\, u'(t) = G(t) = \begin{pmatrix} (t+2)e^{-2t} \\ -3e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} e^{-3t} & 4e^{4t} \\ -2e^{-3t} & -e^{4t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} (t+2)e^{-2t} \\ -3e^{-2t} \end{pmatrix} \Rightarrow \text{(using Cramers rule)}$$

$$u_1'(t) = \frac{\begin{vmatrix} (t+2)e^{-2t} & 4e^{4t} \\ -3e^{-2t} & -e^{4t} \end{vmatrix}}{\begin{vmatrix} e^{-3t} & 4e^{4t} \\ -2e^{-3t} & -e^{4t} \end{vmatrix}} = \frac{e^{2t}\begin{vmatrix} t+2 & 4 \\ -3 & -1 \end{vmatrix}}{e^{t}\begin{vmatrix} 1 & 4 \\ -2 & -1 \end{vmatrix}} =$$

$$= e^{t}\,\frac{-t-2+12}{-1+8} = -\frac{1}{7}e^{t}(t-10)$$ (By the way the denominator here = $7e^{t}$)

$$u_2'(t) = \frac{\begin{vmatrix} e^{-3t} & (t+2)e^{-2t} \\ -2e^{-3t} & -3e^{-2t} \end{vmatrix}}{7e^{t}} = \frac{e^{-5t}}{7e^{t}}\underbrace{\begin{vmatrix} 1 & t+2 \\ -2 & -3 \end{vmatrix}}_{-3+2t+4} = \frac{e^{-6t}}{7}(2t+1)$$

Now $u_1(t) = \int u_1'(t)\,dt = -\frac{1}{7}\int e^{t}(t-10)\,dt \underset{\text{by part}}{=} -\frac{1}{7}\Big(e^{t}(t-10) - \underbrace{\int e^{t}\,dt}_{e^{t}}\Big) =$

$= -\frac{1}{7} e^{t}(t-11) + C_1$ (we can take $C_1 = 0$) $\Rightarrow$

$$u_1(t) = -\frac{1}{7} e^{t}(t-11)$$

$$u_2(t) = \int u_2'(t)\,dt = \frac{1}{7}\int e^{-6t}(2t+1)\,dt \underset{\text{by parts}}{=} \frac{1}{7}\left(-\frac{1}{6}e^{-6t}(2t+1) + \right.$$

$$\left. + \frac{1}{3}\int e^{-6t}\,dt\right) = \frac{1}{7}\left(-\frac{1}{6}e^{-6t}(2t+1) - \frac{1}{18}\cdot e^{-6t}\right) =$$

$$= -\frac{1}{126} e^{-6t}(\underbrace{6t+3+1}_{6t+4}) = -\frac{1}{63}e^{-6t}(3t+2)$$

(again we took the constant of integration here as $C_2 = 0$)

So $x(t) = \Psi(t)u(t) = \begin{pmatrix} e^{-3t} & 4e^{4t} \\ -2e^{-3t} & e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{7}e^{t}(t-11) \\ -\frac{1}{63}e^{-6t}(3t+2) \end{pmatrix} =$

$$= e^{-2t}\begin{pmatrix} -\frac{1}{7}t + \frac{11}{7} - \frac{4}{63}\cdot 3t - \frac{8}{63} \\ \frac{2}{7}t - \frac{22}{7} + \frac{3}{63}t + \frac{2}{63} \end{pmatrix} = e^{-2t}\begin{pmatrix} \left(-\frac{1}{7} - \frac{4}{21}\right)t + \frac{99-8}{63} \\ \left(\frac{2}{7} + \frac{1}{21}\right)t - \frac{198-2}{63} \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} -\frac{7}{21}t + \frac{91}{63} \\ \frac{7}{21}t - \frac{196}{63} \end{pmatrix} = e^{-2t}\begin{pmatrix} -\frac{1}{3}t + \frac{13}{9} \\ \frac{1}{3}t - \frac{28}{9} \end{pmatrix}$$

$\Rightarrow$ the general solution $=$ $\boxed{e^{-2t}\begin{pmatrix} -\frac{1}{3}t + \frac{13}{9} \\ \frac{1}{3}t - \frac{28}{9} \end{pmatrix} + \underbrace{C_1 e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\text{general solution of the homogeneous equation}}}$

Problem 2 (B)

(B) $y'' - 4y' + 4y = \dfrac{e^{2t}}{1+t^2}$ (3) $\Rightarrow g(t) = \dfrac{e^{2t}}{1+t^2}$

1) The corresponding homogeneous equation is

$y'' - 4y' + 4y = 0$ (4) $\Rightarrow$ the characteristic equation is

$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Rightarrow r_{1,2} = 2$

$\Rightarrow \{e^{2t}, te^{2t}\}$ is a fundamental set of solution of (4)

$\Rightarrow$ $\Psi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ (e^{2t})' & (te^{2t})' \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{pmatrix}$ is

the fundamental matrix of the system of first order equations corresponding to 4

2) We look for a solution of (3) in the form

$y(t) = u_1(t)e^{2t} + u_2(t)te^{2t}$ such that

$u_1'(t)$ & $u_2'(t)$ satisfy the system of linear algebraic equations $\Psi(t)\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$ with $g(t) = \dfrac{e^{2t}}{1+t^2}$

$\begin{pmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{e^{2t}}{1+t^2} \end{pmatrix}$ $\Leftrightarrow$ cancelling the common factor e^{2t}

$\begin{pmatrix} 1 & t \\ 2 & 1+2t \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{1+t^2} \end{pmatrix}$ $\Rightarrow$ (using Cramers rule)

$$u_1'(t) = \frac{\begin{vmatrix} 0 & t \\ \frac{1}{1+t^2} & 1+2t \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 1+2t \end{vmatrix}} = \frac{-\frac{t}{1+t^2}}{\underbrace{1+\cancel{2t}-\cancel{2t}}_{\text{Wronskian}}} = -\frac{t}{1+t^2}$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & 0 \\ 2 & \frac{1}{1+t^2} \end{vmatrix}}{\text{Wronskian}} = \frac{\frac{1}{1+t^2}}{1} = \frac{1}{1+t^2}$$

$\Rightarrow u_1(t) = -\int \frac{t}{1+t^2}\,dt = -\frac{1}{2}\ln(1+t^2) + C_1$. Take $C_1 = 0 \Rightarrow$

$$u_1(t) = -\frac{1}{2}\ln(1+t^2)$$

$u_2(t) = \int \frac{dt}{1+t^2} = \arctan t + C_2$. Take $C_2 = 0 \Rightarrow$

$$u_2(t) = \arctan t$$

$\Rightarrow$ a particular solution of (3) is

$$y(t) = u_1(t)e^{2t} + u_2(t)\,te^{2t} = -\frac{1}{2}\ln(1+t^2)e^{2t} + (\arctan t)\,te^{2t}$$

$\Rightarrow$ the general solution of (3) is

$$y(t) = \boxed{-\frac{1}{2}\ln(1+t^2)e^{2t} + (\arctan t)\,te^{2t} + C_1e^{2t} + C_2te^{2t}}$$

Problem 2 (c)

$$X' = \begin{pmatrix} 15 & 8 & -28 \\ 7 & 1 & -10 \\ 9 & 4 & -17 \end{pmatrix} X + \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \\ -2e^{-3t} \end{pmatrix} \qquad (5)$$

$$\Rightarrow \quad G(t) = \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \\ -2e^{-3t} \end{pmatrix}$$

Consider the corresponding homogeneous system

$$X' = \begin{pmatrix} 15 & 8 & -28 \\ 7 & 1 & -10 \\ 9 & 4 & -17 \end{pmatrix} X \qquad (6)$$

1) Find a fundamental set of solutions of (6)

$$A = \begin{pmatrix} 15 & 8 & -28 \\ 7 & 1 & -10 \\ 9 & 4 & -17 \end{pmatrix}$$

- Char. equation : $\det(A - \lambda I) = \begin{vmatrix} 15-\lambda & 8 & -28 \\ 7 & 1-\lambda & -10 \\ 9 & 4 & -17-\lambda \end{vmatrix} = (15-\lambda)\Big(\underbrace{(\lambda-1)(\lambda+17)+40}_{\lambda^2+16\lambda+23}\Big) -$

$$-8\Big(\underbrace{-7(17+\lambda)+90}_{-7\lambda-29}\Big) - 28\Big(\underbrace{28-9+9\lambda}_{19+9\lambda}\Big) = 15\lambda^2 - \lambda^3 + 240\lambda - 16\lambda^2 +$$

$$+ 345 - 23\lambda + 56\lambda + 232 - 532 - 252\lambda = -\lambda^3 - \lambda^2 + 21\lambda + 45$$

Integer roots are divisor of 45, i.e $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15$

$\lambda = 1 \qquad -1 - 1 + 21 + 45 \neq 0 \quad \times$

$\lambda = -1 \qquad 1 - 1 + 21 + 45 \neq 0$

$\lambda = 3 \qquad -27 - 9 + 63 + 45 \neq 0$

$\lambda=-3$ $\quad 27-9-63+45=0 \Rightarrow \lambda=-3$ is a root $\Rightarrow$

$\lambda+3$ divides the characteristic polynomial

$$\begin{array}{r|l} & \lambda^2-2\lambda-15 \\ \hline \lambda+3 & \lambda^3+\lambda^2-21\lambda-45 \\ & \underline{\lambda^3+3\lambda^2} \\ & \quad -2\lambda^2-21\lambda \\ & \quad \underline{-2\lambda^2-6\lambda} \\ & \qquad -15\lambda+45 \end{array}$$

$\lambda^2-2\lambda-15=0$

$D=4+60=64$

$\lambda_2=\frac{2+8}{2}=5$

$\lambda_3=\frac{2-8}{2}=-3$

$\Rightarrow$ $\lambda=-3$ has algebraic multiplicity 2

$\lambda=5$ is a simple nonrepeated eigenvalue

- Eigenspace of $\lambda=-3$

$$(A+3I)v=\begin{pmatrix}18 & 8 & -28\\ 7 & 4 & -10\\ 9 & 4 & -14\end{pmatrix}v=0$$

$$\left(\begin{array}{ccc|c}18 & 8 & -28 & 0\\ 7 & 4 & -10 & 0\\ 9 & 4 & -14 & 0\end{array}\right)\xrightarrow{R_1\to\frac{R_1}{2}}\left(\begin{array}{ccc|c}9 & 4 & -14 & 0\\ 7 & 4 & -10 & 0\\ 9 & 4 & -14 & 0\end{array}\right)\xrightarrow[R_3\to R_3-R_1]{R_2\to 9R_2-7R_1}\left(\begin{array}{ccc|c}9 & 4 & -14 & 0\\ 0 & 8 & 8 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$\Rightarrow$ geometric multiplicity of $\lambda=-3$ is 1

$$\begin{cases}9v_1+4v_2-14v_3=0\\ v_2+v_3=0\end{cases}$$

Using algorithm 2

Let $v_3 = 1 \Rightarrow v_2 = -1 \Rightarrow 9v_1 - 4 - 14 = 0 \Rightarrow v_1 = 2$

$\Rightarrow v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = -3$

(generating the eigenspace)

Find w s.t. $(A + 3I)w = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 18 & 8 & -28 & 2 \\ 7 & 4 & -10 & -1 \\ 9 & 4 & -14 & 1 \end{array}\right) \xrightarrow{R_1 \to \frac{R_1}{2}} \left(\begin{array}{ccc|c} 9 & 4 & -14 & 1 \\ 7 & 4 & -10 & -1 \\ 9 & 4 & -14 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to 9R_2 - 7R_1} \left(\begin{array}{ccc|c} 9 & 4 & -14 & 1 \\ 0 & 8 & 8 & -16 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$9w_1 + 4w_2 - 14w_3 = 1$$
$$w_2 + w_3 = -2$$

Take $w_3 = 0 \Rightarrow w_2 = -2 \Rightarrow 9w_1 - 8 = 1 \Rightarrow w_1 = 1 \Rightarrow$

$w = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \Rightarrow$

$$x^1(t) = e^{-3t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$$

$$x^2(t) = e^{-3t}\left(\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} e^{-3t}(1+2t) \\ -e^{-3t}(2+t) \\ te^{-3t} \end{pmatrix}$$

are solutions of (6)

Using algorithm 1 : Find w s.t $w \neq c\begin{pmatrix}2\\-1\\1\end{pmatrix}$

and $(A+3I)^2 w = 0$

$$(A+3I)^2 = \begin{pmatrix}18 & 8 & -28\\ 7 & 4 & -10\\ 9 & 4 & -14\end{pmatrix}\begin{pmatrix}18 & 8 & -28\\ 7 & 4 & -10\\ 9 & 4 & -14\end{pmatrix} = \begin{pmatrix}324+56-252 & 144+32-112 & -28\cdot4-80\\ 126+28-90 & 56+16-40 & -28-2-40\\ 9\cdot4+28 & 72+16-56 & -4\cdot14-40\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}128 & 64 & -192\\ 64 & 32 & -96\\ 64 & -32 & -96\end{pmatrix}}_{\text{all rows are multiples of one row, as expected}} \Rightarrow (A+3I)^2 w = 0 \Leftrightarrow$$

$\Leftrightarrow (A+3I)^2 w = 0 \Leftrightarrow 64w_1 + 32w_2 - 96w_3 = 0 \Leftrightarrow$

$2w_1 + w_2 - 3w_3 = 0$

We can take $w = \begin{pmatrix}1\\-2\\0\end{pmatrix}$ as before $\Rightarrow v = (A+3I)\begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}18-16\\7-8\\9-8\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

and we obtain the same $x^1(t)$ and $x^2(t)$ as before.

- Let us find an eigenvector of $\lambda = 5$

$$(A-5I)v = \begin{pmatrix}10 & 8 & -28\\ 7 & -4 & -10\\ 9 & 4 & -22\end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c}10 & 8 & -28 & 0\\ 7 & -4 & -10 & 0\\ 9 & 4 & -22 & 0\end{array}\right) \xrightarrow{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{ccc|c}5 & 4 & -14 & 0\\ 7 & -4 & -10 & 0\\ 9 & 4 & -22 & 0\end{array}\right) \xrightarrow[R_3 \to 5R_3 - 9R_1]{R_2 \to 5R_2 - 7R_1} \left(\begin{array}{ccc|c}5 & 4 & -14 & 0\\ 0 & -48 & 48 & 0\\ 0 & -16 & 16 & 0\end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{16}R_3]{R_2 \to \frac{1}{48}R_2} \left(\begin{array}{ccc|c}5 & 4 & -14 & 0\\ 0 & -1 & 1 & 0\\ 0 & -1 & 1 & 0\end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c}5 & 4 & -14 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \Rightarrow \begin{array}{l}5v_1 + 4v_2 - 14v_3 = 0\\ -v_2 + v_3 = 0\end{array}$$

$v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow 5v_1 + 4 - 14 = 0 \Rightarrow v_2 = 2 \Rightarrow$

$\mathbf{z} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 5 \Rightarrow$

$x^3(t) = e^{5t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{5t} \\ e^{5t} \\ e^{5t} \end{pmatrix}$ is a solution of (6)

So $x^1(t), x^2(t), x^3(t)$, where $x^1(t)$ and $x^2(t)$ are found on page 13 form a fundamental set of solutions $\Rightarrow \Psi(t) = \begin{pmatrix} 2e^{-3t} & e^{-3t}(1+2t) & 2e^{5t} \\ -e^{-3t} & -e^{-3t}(2+t) & e^{5t} \\ e^{-3t} & e^{-3t}t & e^{5t} \end{pmatrix}$ is a fundamental matrix

2) We look for a solution of (5) in the form

$x(t) = \Psi(t)\,U(t)$, where $U'(t)$ satisfies

$\Psi(t)\,U'(t) = G(t)$, i.e

$$\begin{pmatrix} 2e^{-3t} & e^{-3t}(1+2t) & 2e^{5t} \\ -e^{-3t} & -e^{-3t}(2+t) & e^{5t} \\ e^{-3t} & e^{-3t}t & e^{5t} \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} 2e^{-3t} \\ e^{-3t} \\ -2e^{-3t} \end{pmatrix}$$

Multiply both parts by e^{3t},

$$\begin{pmatrix} 2 & 1+2t & 2e^{8t} \\ -1 & -(2+t) & e^{8t} \\ 1 & t & e^{8t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1+2t & 2e^{8t} & 2 \\ -1 & -2-t & e^{8t} & 1 \\ 1 & t & e^{8t} & -2 \end{array}\right) \xrightarrow[R_3 \to 2R_3 - R_1]{R_2 \to 2R_2 + R_1} \left(\begin{array}{ccc|c} 2 & 1+2t & 2e^{8t} & 2 \\ 0 & -4-\cancel{2t}+1+\cancel{2t} & 4e^{8t} & 4 \\ 0 & \cancel{2t}-1-\cancel{2t} & 0 & -6 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1+2t & 2e^{8t} & 2 \\ 0 & -3 & 4e^{8t} & 4 \\ 0 & -1 & 0 & -2 \end{array}\right) \Rightarrow \begin{array}{r} 2u_1'(t) + (1+2t)u_2'(t) + 2e^{8t}u_3'(t) = 2 \\ -3u_2'(t) + 4e^{8t}u_3'(t) = 4 \\ u_2'(t) = 6. \end{array}$$

Substituting the 3rd equation into the second:

$$-18 + 4e^{8t}u_3'(t) = 4 \Rightarrow u_3'(t) = \frac{22}{4}e^{-8t} = \frac{11}{2}e^{-8t}$$

Substituting $u_2'(t)$ and $u_3'(t)$ into the 1st equation:

$$2u_1'(t) + (1+2t)\cdot 6 + \cancel{2}e^{8t}\frac{11}{\cancel{2}}e^{-8t} = 2 \Rightarrow$$

$$2u_1'(t) = 2 - 11 - 6 - 12t = -15 - 12t \Rightarrow$$

$$u_1'(t) = -\frac{15+12t}{2} = -\frac{15}{2} - 6t$$

So $u_1(t) = -\frac{15}{2}t - 3t^2 + C_1$ (we can take $C_1 = 0$)

$$u_2(t) = 6t$$

$$u_3(t) = \frac{11}{2}\int e^{-8t}\,dt = -\frac{11}{16}e^{-8t}$$

$\Rightarrow$ A particular solution $x(t) = \Psi(t)u(t) = \begin{pmatrix} 2e^{-3t} & e^{-3t}(1+2t) & 2e^{5t} \\ -e^{-3t} & -e^{-3t}(2+t) & e^{5t} \\ e^{-3t} & e^{-3t}t & e^{5t} \end{pmatrix} \begin{pmatrix} -\frac{15}{2}t - 3t^2 \\ 6t \\ -\frac{11}{16}e^{-8t} \end{pmatrix}$

$$e^{-3t}\begin{pmatrix} -15t - 6t^2 + 6t + 12t^2 - \frac{11}{8} \\ \frac{15}{2}t + 3t^2 - 12t - 6t^2 - \frac{11}{16} \\ -\frac{15}{2}t - 3t^2 + 6t^2 - \frac{11}{16} \end{pmatrix} = e^{-3t}\begin{pmatrix} 6t^2 - 9t - \frac{11}{8} \\ -3t^2 - \frac{9}{2}t - \frac{11}{16} \\ 3t^2 - \frac{15}{2}t - \frac{11}{16} \end{pmatrix}$$

$\Rightarrow$ The general solution is

$$\boxed{x(t) = e^{-3t}\begin{pmatrix} 6t^2 - 9t - \frac{11}{8} \\ -3t^2 - \frac{9}{2}t - \frac{11}{16} \\ 3t^2 - \frac{15}{2}t - \frac{11}{16} \end{pmatrix} + e^{-3t}\left(c_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2 \left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) \right) + c_3 e^{5t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}$$