Homework # 10. Solutions MATH 308 Summer 2012

Problem 1

a) $L = 6\,in = \frac{1}{2}ft$, $w = 4\,lb$

The spring coefficient $k = \frac{w}{L} = \frac{4}{\frac{1}{2}}\frac{lb}{ft} = 8\frac{lb}{ft}$

$m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$

The equation of forced vibrations is

$$\frac{1}{8}u'' + u' + 8u = 4\cos 2t$$

The characteristic equation is

$$\frac{1}{8}r^2 + r + 8 = 0$$

$$D = 1 - 4\cdot\frac{1}{8}\cdot 8 = 1 - 4 = -3$$

$$r_{1,2} = \frac{-1 \pm \sqrt{3}i}{2\cdot\frac{1}{8}} = -4 \pm 4\sqrt{3}\,i$$

$\alpha = 2i$ is not a characteristic root $\Rightarrow$ the multiplicity $s = 0$

We a looking for a particular solution (which will be also the steady state solution) in the form

$$\begin{array}{r|l} 8\times & u(t) = A_1\cos 2t + A_2\sin 2t \Rightarrow \\ 1\times & u' = 2A_2\cos 2t - 2A_1\sin 2t \\ \frac{1}{8}\times & u'' = -4A_1\cos 2t - 4A_2\sin 2t \\ \hline & \frac{1}{8}u'' + u' + 8u = \left(-\frac{1}{2}A_1 + 2A_2 + 8A_1\right)\cos 2t + \left(-2A_1 - \frac{1}{2}A_2 + 8A_2\right)\sin 2t = \end{array}$$

$$= \left(\frac{15}{2}A_1 + 2A_2\right)\cos 2t + \left(-2A_1 + \frac{15}{2}A_2\right)\sin 2t = 4\cos 2t \Rightarrow$$

$$\begin{cases} \frac{15}{2}A_1 + 2A_2 = 4 & \times \frac{15}{2} \\ -2A_1 + \frac{15}{2}A_2 = 0 & \times 2 \end{cases}$$

$$\frac{15}{2}\,Eq\,1 - 2\,Eq\,2 = \frac{225}{4}A_1 + 4A_1 = 30 \Rightarrow$$

$$\frac{241}{4}A_1 = 30 \Rightarrow A_1 = \frac{120}{241}.$$

$$Eq\,1 \Rightarrow \quad 2A_2 = 4 - \frac{15}{2}\cdot\frac{120^{\,60}}{241} = 4 - \frac{900}{241} = \frac{964-900}{241} = \frac{64}{241} \Rightarrow \boxed{A_2 = \frac{32}{241}}$$

$$u_p(t) = \frac{120}{241}\cos 2t + \frac{32}{241}\sin 2t$$

The gen. solution is

$$u(t) = \underbrace{\frac{120}{241}\cos 2t + \frac{32}{241}\sin 2t}_{\text{steady state solution}} + \underbrace{C_1 e^{-4t}\cos 4\sqrt{3}t + C_2 e^{-4t}\sin 4\sqrt{3}t}_{\text{transient solution (tends to 0 as } t \to +\infty)}$$

The answer is $\frac{120}{241}\cos 2t + \frac{32}{241}\sin 2t$

b) In our case $\gamma = 1$, $k = 8$ and $m = \frac{1}{8}$ $\Rightarrow$

$$1 - \frac{\gamma^2}{2km} = 1 - \frac{1}{2\cdot\frac{1}{8}\cdot 8} = 1 - \frac{1}{2} > 0 \Rightarrow$$

$$\omega_{max}^2 = \underbrace{\omega_0^2}_{\frac{k}{m}}\left(1 - \frac{\gamma^2}{2km}\right) = \frac{8}{\frac{1}{8}}\left(1 - \frac{1}{2}\right) = 64\cdot\frac{1}{2} = 32 \Rightarrow \omega_{max} = \sqrt{32}$$

Problem 2 a) The characteristic equation is

$$r^2 - r - 12 = 0 \Rightarrow$$

$$D = 1 + 48 = 49$$

$$r_1 = \frac{1+7}{2} = 4$$

$$r_2 = \frac{1-7}{2} = -3 \quad \Rightarrow$$

e^{4t} and e^{-3t} is a fundamental set of solutions of the homogeneous equation $\Rightarrow$

According to the method of variation of parameters we look for a solution of our equation in the form

$$y(t) = u_1(t)e^{4t} + u_2(t)e^{-3t} \quad \text{where} \begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = g(t) \end{cases}, \text{where}$$

$y_1(t) = e^{4t}$
$y_2(t) = e^{-3t}$
$g(t) = 2e^{-3t}$

$$\Leftrightarrow \begin{cases} e^{4t}u_1' + e^{-3t}u_2' = 0 & \times 3 \\ 4e^{4t}u_1' - 3e^{-3t}u_2' = \underbrace{2e^{-3t}}_{g(t)} \end{cases}$$

$$3\,Eq\,1 + Eq\,2 \Rightarrow 7e^{4t}u_1' = 2e^{-3t} \Rightarrow u_1' = \frac{2}{7}e^{-7t} \Rightarrow$$

$$\Rightarrow u_1 = -\frac{2}{49}e^{-7t} + C_1$$

$$Eq\,1 \Rightarrow \frac{2}{7}e^{4t}e^{-7t} + e^{-3t}u_2' = 0 \Rightarrow \frac{2}{7}e^{-3t} + e^{-3t}u_2' = 0 \Rightarrow u_2' = -\frac{2}{7} \Rightarrow$$

$$u_2 = -\frac{2}{7}t + \tilde{C}_2 \Rightarrow$$

$$y(t) = u_1y_1 + u_2y_2 = \underline{-\frac{2}{49}e^{-3t}} - \frac{2}{7}te^{-3t} + C_1e^{4t} + \underline{\tilde{C}_2e^{-3t}} =$$

$$= \boxed{-\frac{2}{7}te^{-3t} + C_1e^{4t} + C_2e^{-3t}}$$

(here $C_2 = \tilde{C}_2 - \frac{2}{49}$)

b) Char. equation is $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow$
$\cos t$ and $\sin t$ is a fundamendal set of solutions of the hom. eq. $\Rightarrow$ By the method of variation of parameters we look for the solution of our equation in the form

$$y(t) = u_1(t)\cos t + u_2(t)\sin t \quad \text{where}$$

$$\begin{cases} \cos t\, u_1' + \sin t\, u_2' = 0 & \times \cos t \\ -\sin t\, u_1' + \cos t\, u_2' = \dfrac{1}{\cos t} & \times \sin t \end{cases}$$

$$\cos t \times Eq\,1 - \sin t\, Eq\,2 = (\cos^2 t + \sin^2 t)\, u_1' = -\frac{\sin t}{\cos t} = -\tan t \Rightarrow$$

$$u_1' = -\tan t \Rightarrow u_1 = -\int \tan t + C_1 = -\int \frac{\sin t\, dt}{\cos t} + C =$$

$$u = \cos t$$
$$du = -\sin t\, dt$$

$$= \int \frac{du}{u} + C = \ln|u| + C = \ln|\cos t| + C_1 = \ln \cos t + C_1 \text{ for } -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\Rightarrow u_1 = \ln \cos t + C_1$$

$$Eq\,1 \Rightarrow -\cos t \cdot \tan t + \sin t\, u_2' = -\sin t + \sin t\, u_2' = 0 \Rightarrow$$

$$u_2' = 1 \Rightarrow u_2 = t + C_2 \Rightarrow$$

$$y(t) = u_1 y_1 + u_2 y_2 = \boxed{(\ln \cos t) \cos t + t \sin t + C_1 \cos t + C_2 \sin t}$$