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Homework 11, Solutions, MATH 308

Problem 1

1a) $\begin{cases} x_1' = 2x_1 - 5x_2 + \sin t \\ x_2' = x_1 - 2x_2 + \tan t \end{cases}$, $x_1(0) = 0$, $x_2(0) = 0$

Solutions Find a fundamental set of solutions for the homogeneous system

$$\begin{cases} x_1' = 2x_1 - 5x_2 \\ x_2' = x_1 - 2x_2 \end{cases} \qquad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = \lambda^2 - trA\,\lambda + \det A = \lambda^2 + 1 = 0 \Rightarrow$

$trA = 0$

$\det A = -4 + 5 = 1$

$\lambda_{1,2} = \pm i$

Eigenvector of $\lambda = i$:

$$(A - iI)v = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow$$

$v_1 - (2+i)v_2 = 0 \Rightarrow$ If $v_2 = 1$ then $v_1 = 2+i$

$$v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ is an eigenvector $\Rightarrow$

$$e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

is a complex-valued solution

$$\Rightarrow x(t) = C_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$ is

the general solution of the homogeneous system

$\Rightarrow$ the fundamental matrix $\varphi(t)$ can be taken as

$$\varphi(t) = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}$$

and a solution of the original system is found in the form

$$x(t) = u_1(t) \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + u_2(t) \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

s.t $$\varphi(t) \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \sin t \\ \tan t \end{pmatrix}}_{G(t)}$$

(using Cramer's rule)

$$\Rightarrow u_1'(t) = \frac{\begin{vmatrix} \sin t & 2\sin t + \cos t \\ \tan t & \sin t \end{vmatrix}}{\begin{vmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{vmatrix}} = \frac{\sin^2 t - 2\sin t \tan t - \sin t}{\underbrace{\cancel{2\cos t \sin t} - \sin^2 t - \cancel{2\sin t \cos t} - \cos^2 t}_{-1}}$$

$$= -\sin^2 t + 2\frac{\overbrace{\sin^2 t}^{1-\cos^2 t}}{\cos t} + \sin t = \overbrace{-\sin^2 t}^{-\frac{1-\cos 2t}{2}} + 2\sec t - 2\cos t + \sin t$$

$$u_1(t) = -\frac{1}{2}t + \frac{\sin 2t}{4} + 2\ln|\sec t + \tan t| - 2\sin t - \cos t + C_1$$

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$$u_2'(t) = \frac{\begin{vmatrix} 2\cos t - \sin t & \sin t \\ \cos t & \tan t \end{vmatrix}}{\underbrace{-1}_{\det \Psi(t) \text{ calculated before}}} = -\left(2\sin t - \frac{\sin^2 t}{\cos t} - \sin t\cos t\right)$$

$$= -2\sin t + \frac{1-\cos^2 t}{\cos t} + \frac{1}{2}\sin 2t = -2\sin t + \sec t - \cos t +$$

$$+ \frac{1}{2}\sin 2t \Rightarrow u_2(t) = 2\cos t + \ln|\sec t + \tan t| -$$

$$-\sin t - \frac{1}{4}\cos 2t + C_2 \Rightarrow$$

The general solution of the original system is

$$x(t) = u_1(t)\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + u_2(t)\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} =$$

$$= \left(-\frac{1}{2}t + \frac{\sin 2t}{4} + 2\ln|\sec t + \tan t| - 2\sin t - \cos t + C_1\right) \times$$

$$\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \left(2\cos t + \ln|\sec t + \tan t| - \sin t - \frac{1}{4}\cos 2t +\right.$$

$$\left. + C_2\right) \cdot \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (C_1 - 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \underbrace{\left(2 - \frac{1}{4} + C_2\right)}_{C_2 + \frac{7}{4}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow c_1 = 1, \quad c_2 = -\frac{7}{4}$ (because $\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$)

$$X(t) = \left(-\frac{1}{2}t + \frac{\sin 2t}{4} + 2\ln|\sec t + \tan t| - 2\sin t - \cos t + 1\right)\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} +$$
$$\left(2\cos t + \ln|\sec t + \tan t| - \sin t + \frac{1}{4}\cos 2t - \frac{7}{4}\right)\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

Componentwise:

- $\boxed{x_1(t)} = -t\cos t + \frac{1}{2}t\sin t + \frac{1}{2}\sin 2t\cos t - \frac{1}{4}\sin 2t \sin t +$
$+ (4\cos t - 2\sin t)\ln|\sec t + \tan t| - \cancel{4\sin t\cos t} + \cancel{2\sin^2 t} - \cancel{2\cos^2 t} +$
$\cancel{\sin t\cos t} + 2\cos t - \sin t + \cancel{4\sin t\cos t} + \cancel{2\cos^2 t} + (2\sin t + \cos t)\ln|\sec t + \tan t|$
$- \cancel{2\sin^2 t} - \cancel{\sin t\cos t} - \frac{1}{2}\sin t\cos 2t - \frac{1}{4}\cos 2t\cos t - \frac{7}{2}\sin t - \frac{7}{4}\cos 2t =$
$= -t\cos t + \frac{1}{2}t\sin t + \frac{1}{2}(\underbrace{\sin 2t\cos t - \sin t\cos 2t}_{\sin(2t-t) = \sin t}) \cancel{- \frac{1}{4}(\underbrace{\sin 2t\sin t + \cos 2t\cos t}_{\cos(2t-t)})}$
$+ \cancel{\frac{1}{2}\cos t} - \frac{9}{2}\sin t + 5\cos t\ln|\sec t + \tan t| = \boxed{-t\cos t + \frac{1}{2}t\sin 2t + 5\cos t\ln|\sec t + \tan t| - 4\sin t}$

- $\boxed{x_2(t)} = -\frac{1}{2}t\cos t + \frac{1}{4}\sin 2t\cos t + 2\cos t\ln|\sec t + \tan t| -$
$- 2\sin t\cancel{\cos t} - \cos^2 t + \cos t + 2\cancel{\cos t}\sin t + \sin t\ln|\sec t + \tan t| -$
$- \sin^2 t + \frac{1}{4}\cos 2t\sin t - \frac{7}{4}\sin t = \boxed{-\frac{1}{2}t\cos t + \frac{1}{4}\sin 3t - 1 + (2\cos t + \sin t)\ln|\sec t + \tan t| + \cos t - \frac{7}{4}\sin t}$

$$\boxed{x(t)=\begin{pmatrix}-t\cos t+\frac{1}{2}t\sin 2t+5\cos t\ln|\sec t+\tan t|-4\sin t\\ -\frac{1}{2}t\cos t+\frac{1}{4}\sin 3t-1+(2\cos t+\sin t)\ln|\sec t+\tan t|+\cos t-\frac{7}{4}\sin t\end{pmatrix}}$$

Problem 1 (b)

$$2y''-3y'+y=e^t(1+t^2),\quad y(0)=3,\ y'(0)=-4$$

$\Leftrightarrow$ (divide by 2) $\quad y''-\frac{3}{2}y'+\frac{1}{2}y=\frac{1}{2}e^t(1+t^2) \Rightarrow \boxed{g(t)=\frac{1}{2}e^t(1+t^2)}$

1) Find a fundamental set of solutions of the corresponding homogeneous equation.

Characteristic equation: $2r^2-3r+1=0$

$$D=9-8=1$$

$$r_1=\frac{3-1}{4}=\frac{1}{2}$$

$$r_2=\frac{3+1}{4}=1$$

$\Rightarrow$ general solution of the homogeneous equation is

$$y(t)=c_1e^{\frac{1}{2}t}+c_2e^t \text{ \& } y_1(t)=e^{\frac{1}{2}t},\ y_2(t)=e^t \text{ is a fundamental set of solutions}$$

We look for a solution of the original nonhomogeneous equation in the form

$$y(t)=u_1(t)e^{\frac{1}{2}t}+u_2e^t \text{ s.t}$$

$$\begin{pmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{pmatrix}\begin{pmatrix}u_1'(t)\\ u_2'(t)\end{pmatrix}=\begin{pmatrix}0\\ g(t)\end{pmatrix} \Leftrightarrow \begin{pmatrix}e^{\frac{1}{2}t} & e^t\\ \frac{1}{2}e^{\frac{1}{2}t} & e^t\end{pmatrix}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=\begin{pmatrix}0\\ \frac{1}{2}e^t(1+t^2)\end{pmatrix}$$

* Again to get the right g you must to divide the equation by the coefficient of y''.

$$\Rightarrow \quad u_1' = -\frac{g\, y_2(t)}{Wr(y_1,y_2)(t)}, \quad u_2' = \frac{g\, y_1(t)}{Wr(y_1,y_2)(t)}$$

$$Wr(y_1,y_2)(t) = \begin{vmatrix} e^{\frac{1}{2}t} & e^{t} \\ \frac{1}{2}e^{\frac{1}{2}t} & e^{t} \end{vmatrix} = e^{\frac{3}{2}t} - \frac{1}{2}e^{\frac{3}{2}t} = \frac{1}{2}e^{\frac{3}{2}t}$$

$$u_1' = -\frac{\cancel{\frac{1}{2}} e^{t}(1+t^2)e^{t}}{\cancel{\frac{1}{2}} e^{\frac{3}{2}t}} = -(1+t^2)e^{t/2} \Rightarrow$$

$$u_1 = -\int (1+t^2)e^{t/2}dt \underset{\substack{\downarrow \\ \text{integration} \\ \text{by parts}}}{=} -\left(2(1+t^2)e^{t/2} - 4\int te^{t/2}\right) =$$

$$= -\left(2(1+t^2)e^{t/2} - 8te^{t/2} + 8\int e^{t/2}dt\right) =$$

$$= -(2t^2+2-8t+16)e^{t/2} + C_1 = -2(t^2-4t+9)e^{t/2} + C_1$$

$$u_2' = \frac{\frac{1}{2}\cancel{e^{t}}(1+t^2)\cancel{e^{t/2}}}{\frac{1}{2}\cancel{e^{3/2t}}} = (1+t^2) \Rightarrow$$

$$u_2 = \int (1+t^2)dt = \frac{t^3}{3} + t + C \Rightarrow$$

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -2(t^2-4t+9)e^{t} +$$

$$+ \left(\frac{t^3}{3} + t\right)e^{t} + C_1 e^{t/2} + C_2 e^{t} =$$

$$= \left(\frac{t^3}{3} - 2t^2 + 9t - 18\right)e^t + c_1 e^{t/2} + c_2 e^t$$

$y(0) = 3 \Rightarrow \quad -18 + c_1 + c_2 = 3 \Rightarrow \quad c_1 + c_2 = 21$

$y'(t) = \left(t^2 - 4t + 9\right)e^t + \left(\frac{t^3}{3} - 2t^2 + 9t - 18\right)e^t +$

$+ \frac{1}{2} c_1 e^{t/2} + c_2 e^t \Rightarrow$

$y'(0) = -9 + \frac{1}{2}c_1 + c_2 = -4 \Rightarrow \frac{1}{2}c_1 + c_2 = 5$

$$\begin{cases} c_1 + c_2 = 21 \\ \frac{1}{2}c_1 + c_2 = 5 \end{cases} \xrightarrow{Eq1 - Eq2} \frac{1}{2}c_1 = 16 \Rightarrow c_1 = 32 \Rightarrow$$

$$c_2 = 21 - c_1 = -11$$

$$\Rightarrow y(t) = \left(\frac{t^3}{3} - 2t^2 + 9t - 18 - 11\right)e^t + 32e^{t/2} =$$

$$= \boxed{\left(\frac{t^3}{3} - 2t^2 + 9t - 29\right)e^t + 32e^{t/2}}$$

Problem 1 (c)

$$X' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t, \quad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

First find a fundamental set of solutions of the corresponding homogeneous system

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$$X' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} X$$

- Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} =$$

expanding w.r.t. the first column

$$= (2-\lambda)(2-\lambda)(3-\lambda) = 0 \Rightarrow$$

$\lambda_1 = 2$ with algebraic multiplicity 2

$\lambda_2 = 3$ with algebraic multiplicity 1

- $\lambda = 2$: Eigenspace: $(A - 2I)v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} v = 0$

$\Rightarrow$ geom. multiplicity is 1: $\begin{matrix} v_3 = 0 \\ v_2 + v_3 = 0 \end{matrix} \Rightarrow v_2 = v_3 = 0 \Rightarrow$

$$E_2 = \left\{ \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}, c \in \mathbb{R} \right\}$$

For example we can take $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Find a generalized eigenvector w s.t. $(A - 2I)w = v$

i.e $\begin{pmatrix}0&0&1\\0&0&0\\0&1&1\end{pmatrix} w = \begin{pmatrix}1\\0\\0\end{pmatrix} \iff$ $w_3 = 1$

$w_2 + w_3 = 0 \Rightarrow w_2 = -1$

$\Rightarrow w = \left\{ \begin{pmatrix}c\\-1\\1\end{pmatrix}, c \in \mathbb{R} \right\}$. We can take $c = 0$, i.e.

$w = \begin{pmatrix}0\\-1\\1\end{pmatrix} \Rightarrow$

So $x^1(t) = e^{2t} \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v} = \begin{pmatrix}e^{2t}\\0\\0\end{pmatrix}$ and $x^2(t) = e^{2t}\left(\underbrace{\begin{pmatrix}0\\-1\\1\end{pmatrix}}_{v} + \underbrace{t\begin{pmatrix}1\\0\\0\end{pmatrix}}_{+t\,w}\right) =$

$= \begin{pmatrix}te^{2t}\\-e^{2t}\\e^{2t}\end{pmatrix}$ are independent solutions

$\lambda = 3$ Find an eigenvector:

$(A - 3I)v = 0 \iff \begin{pmatrix}-1&0&1\\0&-1&0\\0&1&0\end{pmatrix} v = 0$

$\left(\begin{array}{ccc|c}-1&0&1&0\\0&-1&0&0\\0&1&0&0\end{array}\right) \underset{R_2 \to -R_2}{\overset{R_3 \to R_3 - R_2}{\sim}} \left(\begin{array}{ccc|c}-1&0&1&0\\0&1&0&0\\0&0&0&0\end{array}\right) \iff \begin{matrix}-v_1 + v_3 = 0\\ v_2 = 0\end{matrix} \Rightarrow$

Take $v_3 = 1 \Rightarrow v_1 = 1 \Rightarrow v = \begin{pmatrix}1\\0\\1\end{pmatrix} \Rightarrow$

$x^3(t) = e^{3t}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}e^{3t}\\0\\e^{3t}\end{pmatrix}$ is a solution

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- Putting all three obtained solutions in columns we get the fundamental matrix

$$\Upsilon(t) = \begin{pmatrix} e^{2t} & te^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{pmatrix}$$

- We look for a solution of the original nonhomogeneous equation in the form

$$x(t) = u_1(t)\, x^1(t) + u_2(t)\, x^2(t) + u_3(t)\, x^3(t) \quad \text{s.t}$$

$$\Upsilon(t) \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{t} \\ 0 \\ e^{t} \end{pmatrix}}_{G}$$

Clearly from the expression for $\Upsilon(t)$ we get that $u_1'(t) = u_2'(t) = 0$ and $u_3'(t) = e^{-2t}$

(The third column of the matrix $\Upsilon(t)$ is colinear to the column vector $G(t)$ with the coefficient of proportionality e^{-2t})

$\Rightarrow u_1(t) = C_1, \ u_2(t) = C_2, \ u_3(t) = -\frac{1}{2}e^{-2t} + C_2$

$\Rightarrow$ the general solution of our system is

$$X(t) = -\frac{1}{2}\begin{pmatrix} e^t \\ 0 \\ e^t \end{pmatrix} + C_1\begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} te^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix} + C_3\begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \\ 3/2 \end{pmatrix} \Leftrightarrow$$

$$\left.\begin{cases} C_1 + C_3 = 3/2 \\ -C_2 = 1 \\ C_2 + C_3 = 3/2 \end{cases}\right\} \Rightarrow \begin{array}{l} C_1 + 5/2 = 3/2 \Rightarrow C_1 = -1 \\ C_2 = -1 \\ -1 + C_3 = 3/2 \Rightarrow C_3 = 5/2 \end{array}$$

$$\Rightarrow X(t) = -\frac{1}{2}\begin{pmatrix} e^t \\ 0 \\ e^t \end{pmatrix} - \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} te^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix} + \frac{5}{2}\begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{2}e^t - e^{2t} - te^{2t} + \frac{5}{2}e^{3t} \\ e^{2t} \\ -\frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t} \end{pmatrix}$$

Problem 2

(a) i $$y'' + 3y' - 18y = \underbrace{t^4}_{g_1(t)} + \underbrace{(t^2+3)e^{-t}}_{g_2(t)} + \underbrace{te^{-6t}}_{g_3(t)} + \underbrace{(t^4+t^2)e^{3t}}_{g_4(t)} \quad (1)$$

Root of the characteristic equation: $r^2+3r-18=0$

$D = 9+72 = 81$

$r_1 = \frac{-3+9}{2} = \boxed{3}$

$r_2 = \frac{-3-9}{2} = \boxed{-6}$

The right-hand side of (1) is the sum of 4 terms such that each of them is of the form for the applying the method of undetermined coefficients. Consider 4 equations with the same left-hand side as in (1) and left-hand side g_1, g_2, g_3, or g_4 separately and then add the resulting solutions:

1) $$y'' + 3y' - 18y = \underbrace{t^4}_{e^{0t}P_4(t)} \quad (1a)$$

$P_4(t)$ — a polynomial of degree 4

So $\alpha = 0$ and it is not a root of the characteristic equation $\Rightarrow s=0$

$\Rightarrow$ A solution of (1a) can be found in the form

$$y_1(t) = A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0$$

2) $$y'' + 3y' - 18y = \underbrace{(t^2+3)}_{\text{a polynomial of degree 2}}e^{-t} \quad (1b)$$

$\alpha = -1$ is not a root of the characteristic equation $\Rightarrow s=0$

$\Rightarrow$ A solution of (1b) can be found in the form

$$y_2(t) = e^{-t}(B_2t^2 + B_1t + B_0)$$

3) $y'' + 3y' - 18y = \underbrace{t}_{\text{a polynomial of degree 1}} e^{-6t}$ (1c) $\rightarrow$ $\alpha = -6$ is a root but not a repeated root of the characteristic equation $\Rightarrow s = 1$

$\Rightarrow$ A solution of (1c) can be found in the form

$$y_3(t) = \underbrace{t}_{t^s} e^{-6t}(C_1 t + C_0)$$

4) $y'' + 3y' - 18y = (t^4 + t^3) e^{3t}$ (1d) $\rightarrow$ $\alpha = 3$ is a root but not a repeated root of the characteristic equation $\Rightarrow s = 1$

a polynomial of degree 4

$\Rightarrow$ A solution of (1d) can be found in the form

$$y_4(t) = \underbrace{t}_{t^s} e^{3t}(D_4 t^4 + D_3 t^3 + D_2 t^2 + D_1 t + D_0)$$

Summing up all solutions from 1)-4) we get that a particular solution of the original equation can be found in the form

$$\boxed{\begin{aligned} y(t) = {} & A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0 + e^{-t}(B_2 t^2 + B_1 t + B_0) + \\ & + t e^{-6t}(C_1 t + C_0) + t e^{3t}(D_4 t^4 + D_3 t^3 + D_2 t^2 + D_1 t + D_0) \end{aligned}}$$

Problem 2 1(a) ii

$y''+10y'+25y = e^{-5t} + (t+1)e^{3t}$

Roots of the characteristic equation: $r^2+10r+25=0 \iff (r+5)^2=0 \iff$
$r_1=r_2=\boxed{-5}$

1) $y''+10y'+25y = e^{-5t} \longrightarrow$ (2a) $\Rightarrow \alpha=-5$

degree of the polynomial $=0$

$\alpha=-5$ is the repeated root of the characteristic equation $\Rightarrow$
$\underline{s=2}$ $\Rightarrow$ a particular solution is found in the form
$y_1(t) = At^2e^{-5t}$

2) $y''+10y'+25y = \underbrace{(t+1)}_{\text{a polynomial of degree 1}}e^{3t} \longrightarrow$ (2b) $\alpha=3$ is not a root of the characteristic polynomial $\Rightarrow$
$s=0$

$\Rightarrow$ a particular solution is found in the form
$y_2(t) = (Bt+C)e^{3t}$

$\Rightarrow$ combining 1) & 2) a particular solution of the original equation is found in the form

$$\boxed{y(t) = At^2e^{-5t} + (Bt+C)e^{3t}}$$

Problem 2 b) Now we want to find A, B, & C

1) To find A plug $y(t) = At^2e^{-5t}$ into (2a)

$25 \times \quad y(t) = A t^2 e^{-5t}$

$10 \times \quad y'(t) = -5At^2e^{-5t} + 2Ate^{-5t}$

$+$

$y''(t) = 25At^2e^{-5t} - 10Ate^{-5t} - 10Ate^{-5t} + 2Ae^{-5t} =$

$= 25At^2e^{-5t} - 20Ate^{-5t} + 2Ae^{-5t}$

$$y'' + 10y' + 25y = (\underbrace{25A - 50A + 25A}_{0})t^2e^{-5t} + (\underbrace{-20A + 20A}_{0})te^{-5t} + 2Ae^{-5t} \stackrel{!}{=} e^{-5t}$$

$\Rightarrow 2A = 1 \Rightarrow \boxed{A = \frac{1}{2}}$

2) To find B & C plug $y(t) = (Bt + C)e^{3t}$ into (2b)

$25 \times \quad y(t) = (Bt + C)e^{3t}$

$10 \times \quad y'(t) = 3(Bt + C)e^{3t} + Be^{3t} = (3Bt + 3C + B)e^{3t}$

$y''(t) = 9(Bt + C)e^{3t} + 3Be^{3t} + 3Be^{3t} = (9Bt + 9C + 6B)e^{3t}$

$$y'' + 10y' + 25y = \left[(\underbrace{9B + 30B + 25B}_{64B})t + (\underbrace{6B + 10B}_{16B} + \underbrace{9C + 30C + 25C}_{64C})\right]e^{3t} =$$

$$= (64Bt + (16B + 64C))e^{3t} = (t + 1)e^{3t} \Rightarrow \text{comparing coefficients}$$

$\begin{cases} 64B = 1 \Rightarrow \boxed{B = \frac{1}{64}} \\ 16B + 64C = 1 \Rightarrow \overbrace{16 \cdot \frac{1}{64}}^{1/4} + 64C = 1 \Rightarrow 64C = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow C = \frac{3}{256} \Rightarrow \end{cases}$

Plugging A, B, C into the answer of problem 2 a ii we have that the general solution is:

$$\boxed{y(t) = \frac{1}{2}t^2e^{-5t} + \left(\frac{1}{64}t + \frac{3}{256}\right)e^{3t} + \underbrace{C_1e^{-5t} + C_2te^{-5t}}_{\text{gen. sol. of hom. equation}}}$$