Homework #12 - Solutions MATH 308-505 FALL 2012

Problem 1

$$y''+9y=2\delta(t-\tfrac{\pi}{2})-\delta(t-\tfrac{3\pi}{2}),\ y(0)=1,\ y'(0)=0$$

Solution Apply Laplace transform to both sides:

$$\mathcal{L}\{y''\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s$$

$$\mathcal{L}\{\delta(t-\tfrac{\pi}{2})\}=e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}=e^{-\frac{3\pi}{2}s}$$

$$\Downarrow$$

$$(s^2+9)Y(s)-s=2e^{-\frac{\pi}{2}s}-e^{-\frac{3\pi}{2}s}\Rightarrow$$

$$Y(s)=\frac{s}{s^2+9}+2e^{-\frac{\pi}{2}s}\frac{1}{s^2+9}-e^{-\frac{3\pi}{2}s}\frac{1}{s^2+9}$$

1. $\mathcal{L}^{-1}\{Y(s)\}=\cos 3t$
2. $\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}\frac{1}{s^2+9}\}$?

First $\mathcal{L}^{-1}\{\frac{1}{s^2+9}\}=\frac{1}{3}\mathcal{L}^{-1}\{\frac{3}{s^2+9}\}=\frac{1}{3}\sin 3t$

$$\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}\frac{1}{s^2+9}\}=\frac{u_{\frac{\pi}{2}}(t)}{3}\sin 3(t-\tfrac{\pi}{2})=\frac{u_{\frac{\pi}{2}}(t)}{3}\sin(3t-\tfrac{3\pi}{2})=$$

$$=-\frac{1}{3}u_{\frac{\pi}{2}}(t)\sin(\tfrac{3\pi}{2}-3t)=\frac{1}{3}u_{\frac{\pi}{2}}(t)\cos 3t$$

$$\mathcal{L}^{-1}\{e^{-\frac{3\pi}{2}}\frac{1}{s^2+9}\}=\frac{1}{3}u_{\frac{3\pi}{2}}(t)\sin 3(t-\tfrac{3\pi}{2})=\frac{1}{3}u_{\frac{3\pi}{2}}(t)(3t-\tfrac{9\pi}{2})=$$

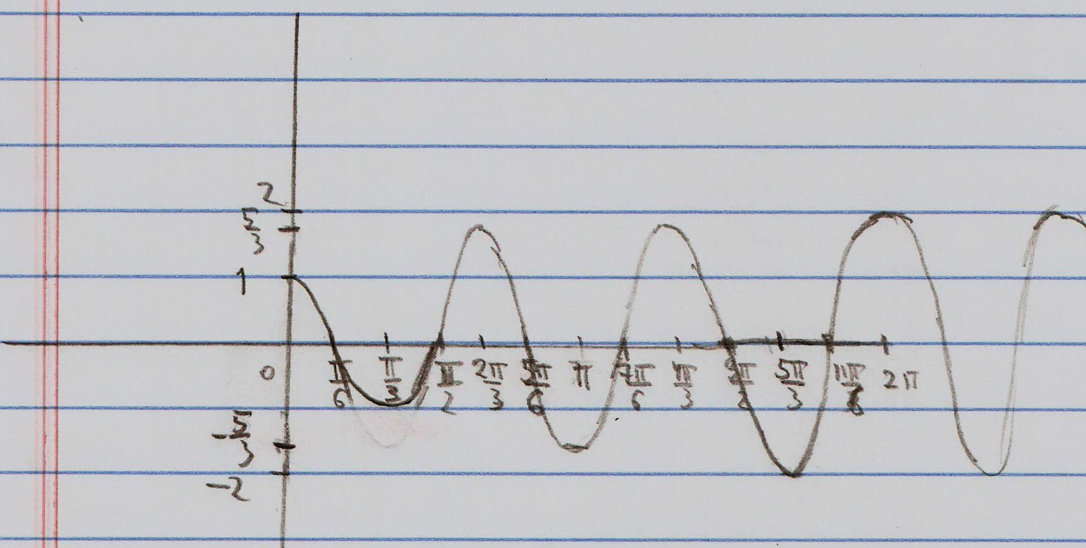
$$=-\frac{1}{3}u_{\frac{3\pi}{2}}(t)\sin(\tfrac{9\pi}{2}-3t)=-\frac{1}{3}u_{\frac{3\pi}{2}}(t)\sin(\tfrac{\pi}{2}-3t)=-\frac{1}{3}u_{\frac{3\pi}{2}}(t)\cos 3t\Rightarrow$$

$$\Downarrow$$

$$y(t) = \boxed{\cos 3t + \frac{2}{3}u_{\frac{\pi}{2}}(t)\cos 3t + \frac{1}{3}u_{\frac{3\pi}{2}}(t)\cos 3t}$$

To sketch $y(t)$ note that

$$y(t) = \begin{cases} \cos 3t & 0 \le t < \frac{\pi}{2} \\ \frac{5}{3}\cos 3t & \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ 2\cos 3t & t \ge \frac{3\pi}{2} \end{cases}$$



<u>Problem 2</u> $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$ using convolution formula

Let $F(s) = \frac{s}{s^2+1}$, $G(s) = \frac{1}{s^2+4} \Rightarrow$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos t, \quad g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g\,(t) = \frac{1}{2}\int_0^t \underbrace{\cos(t-\tau)\sin 2\tau}_{\frac{1}{2}(\sin(2\tau+(t-\tau)) + \sin(2\tau-t+\tau))}\,d\tau =$$

$$= \frac{1}{4}\int_0^t \sin(t+\tau) + \sin(3\tau - t)\,dt =$$

$$= -\frac{1}{4}\cos(t+\tau)\Big|_0^t - \frac{1}{12}\cos(3\tau - t)\Big|_0^t = -\frac{1}{4}\cos 2t + \frac{1}{4}\cos t -$$

$$-\frac{1}{12}\cos 2t + \frac{1}{12}\underbrace{\cos(-t)}_{\cos t} = \left(\frac{1}{4}+\frac{1}{12}\right)\cos t - \left(\frac{1}{12}+\frac{1}{4}\right)\cos 2t =$$

$$= \boxed{\frac{1}{3}\left(\cos t - \cos 2t\right)}$$

Problem 3

(a) $y'' - 4y' + 20y = g(t)$, $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s,$$

$G(s) = \mathcal{L}\{g(t)\}$

$$(s^2 - 4s + 20)Y(s) - s + 4 = G(s) \Rightarrow$$

$$Y(s) = \frac{s-4}{s^2-4s+20} + \frac{G(s)}{s^2-4s+20}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\} = \frac{1}{4}e^{2t}\sin 4t$$

$$\frac{s-4}{s^2-4s+20} = \frac{s-2-2}{(s-2)^2+16} = \frac{s-2}{(s-2)^2+16} - \frac{1}{2}\frac{4}{(s-2)^2+16} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s-4}{s^2-4s+20}\right\} = e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t$$

$$y(t) = e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t + \frac{1}{4}\int_0^T e^{2(t-\tau)}\sin 4(t-\tau)\, g(\tau)\, d\tau$$

(B) Use the method of variation of parameter:

Hom. eq $\quad y'' - 4y' + 20y = 0$

Char. eq $\quad r^2 - 4r + 20 = 0$

$D = 16 - 80 = -64$

$r_1 = \frac{4 \pm 8i}{2} = 2 \pm 4i \Rightarrow$

gen. solution of hom. eq. is

$y(t) = C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t$

We look for the solutions of our original equation in the form

$y(t) = u_1(t) e^{2t}\cos 4t + u_2(t) e^{2t}\sin 4t$ s.t.

$$\begin{cases} e^{2t}\cos 4t\, u_1'(t) + e^{2t}\sin 4t\, u_2'(t) = 0 & \text{Eq 1} \\ (2e^{2t}\cos 4t - 4e^{2t}\sin 4t) u_1'(t) + (2e^{2t}\sin 4t + 4e^{2t}\cos 4t) u_2'(t) = g(t) & \text{Eq 2} \end{cases}$$

Eq (2) -2 Eq 1

$$\begin{cases} e^{2t}\cos 4t\, u_1'(t) + e^{2t}\sin 4t\, u_2'(t) = 0 \\ -4e^{2t}\sin 4t\, u_1'(t) + 4e^{2t}\cos 4t\, u_2'(t) = g \end{cases}$$

By the Cramer rule

$$u_1' = \frac{-g e^{2t}\sin 4t}{\begin{vmatrix} e^{2t}\cos 4t & e^{2t}\sin 4t \\ -4e^{2t}\sin 4t & 4e^{2t}\cos 4t \end{vmatrix}} = -\frac{g e^{2t}\sin 4t}{4e^{4t}} = -\frac{g \sin 4t}{4}$$

$$u_2' = \frac{g e^{2t}\cos 4t}{4e^{4t}} = \frac{g\cos 4t}{4e^{2t}}$$

$$y(t) = \int_0^t \left(-e^{2t}\cos 4t \frac{\sin 4\tau}{4e^{2\tau}} g(\tau) + e^{2t}\sin 4t \frac{\cos 4\tau}{4e^{2\tau}} g(\tau)\right) d\tau +$$

$+ C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t =$

$= \frac{1}{4}\int e^{2(t-\tau)}\underbrace{(\sin 4t\cos 4\tau - \cos 4t\sin 4\tau)}_{\sin 4(t-\tau)}\, g(\tau)\, d\tau +$

$+ C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t$

$y(0) = 1 \Rightarrow C_1 = 1$

$y'(0) = 0 \Rightarrow 2C_1 e^{2t}\cos 4t - 4C_1 e^{2t}\sin 4t + 2C_2 e^{2t}\sin 4t + 4C_2 e^{2t}\cos 4t\Big|_{t=0} =$

$= 2C_1 + 4C_2 = 0 \Rightarrow C_2 = -\frac{1}{2}C_1 = -\frac{1}{2} \Rightarrow$

$y(t) = \frac{1}{4}\int e^{2(t-\tau)}\sin 4(t-\tau)\, d\tau + e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t$

$\Rightarrow$ the answer coincides with item a)