Homework Assignment 11 in Differential Equations, MATH308-Spring 2017, Regular section

Bonus homework. Problem 2 is a good practice for the final exam (as an example of qualitative analysis of nonlinear systems). We did not discuss the Lotka-Volterra equations and you do not need to study the specifics about them from section 9.5 for the final exam (but the latter will be helpful in Problem 2). Also, there will not be problems similar to problem 1 in the final exam but it is still a good practice problem for the case of repeated eigenvalues (as a mixture of complex eigenvalues and repeated eigenvalues).

due May 9 at the beginning of the final exam

Topics covered : *Complex repeated root (a mixture of the material of sections 7.6 and 7.8); predator-prey model (Lottka-Volterra equation) (section 9.5) with some modification (self-limiting terms).*

1. Consider the following system of linear differential equations:

$$\begin{cases} x_1' &= 7x_1 + 3x_2 + 5x_3 + 7x_4 \\ x_2' &= -7x_1 - 3x_2 - 17x_3 - x_4 \\ x_3' &= 2x_1 + 6x_2 + 10x_3 - 4x_4 \\ x_4' &= -2x_1 - 6x_2 + 2x_3 + 10x_4 \end{cases}$$

It is known that the characteristic polynomial of the matrix of the system is equal to $(\lambda^2 - 12\lambda + 72)^2$ and , in particular, it has eigenvalues $6+6i$ and $6-6i$. It is also known that $6+6i$ has an eigenvector $(1-i, 1, -1, 1+i)^T$. Find the general solution of the system.

2. Consider the following system of differential equations

$$\begin{cases} x' &= x(a - \sigma x - \alpha y) \\ y' &= y(-c + \gamma x). \end{cases},$$

where a, σ, α, c, and γ are positive constants with $\sigma < \dfrac{a\gamma}{c}$. This is a modification of the Lotka-Volterra equation from section 9.5 (see equation (1) on page 545 there): the right-hand side of the first equation contains an additional term $-\sigma x^2$, which is responsible for self-limiting of the prey population.

   (a) Find all critical points of the given system. How does their location change as σ increases from 0? Why the assumption $\sigma < \dfrac{a\gamma}{c}$ is natural from the point of view of the predator-prey model?

   (b) Classify all critical points and determine their stability properties (under assumption that $0 < \sigma < \dfrac{a\gamma}{c}$). In particular, show that there is a value of σ between 0 and $\dfrac{a\gamma}{c}$ where the critical point in the interior of the first quadrant changes from a spiral point to a node.

   (c) Based on your answer in the previous item describe the effect on the two populations as σ increases and for all possible situations sketch the phase portrait in the first quadrant.