

# Homework 11 - MATH 308 - Summer 20R - Solutions

## Problem 1

$$a) \mathcal{L}\{\cosh bt\} = \mathcal{L}\left\{\frac{e^{bt} + e^{-bt}}{2}\right\} = \frac{1}{2}(\mathcal{L}\{e^{bt}\} + \mathcal{L}\{e^{-bt}\}) = \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right) = \frac{1}{2} \frac{s+b+s-b}{s^2-b^2} = \frac{1}{2} \frac{2s}{s^2-b^2} = \frac{s}{s^2-b^2}, \quad s > |b|$$

$$b) \mathcal{L}\{e^{at} \sinh bt\}$$

First find  $\mathcal{L}\{\sinh bt\}$ :

$$\text{Indeed } \mathcal{L}\{\sinh bt\} = \mathcal{L}\left\{\frac{e^{bt} - e^{-bt}}{2}\right\} = \frac{1}{2}(\mathcal{L}\{e^{bt}\} - \mathcal{L}\{e^{-bt}\}) = \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{1}{2} \frac{s+b-(s-b)}{s^2-b^2} = \frac{b}{s^2-b^2}$$

$$\Downarrow$$

$$\mathcal{L}\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2} \quad s-a > |b|$$

## Problem 2

$$a) y'' - 5y' + 4y = e^{-3t} \sin 4t, \quad y(0) = 0, y'(0) = -1$$

Apply Laplace transform:

$$4 \times \quad \mathcal{L}\{4y\} = Y(s)$$

$$-5 \times \quad \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$$

$$1 \times \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - (-1) = s^2Y(s) + 1$$

$$\mathcal{L}\{y'' - 5y' + 4y\} = (s^2 - 5s + 4)Y(s) + 1 = \mathcal{L}\{e^{-3t} \sin 4t\} = \frac{4}{(s+3)^2 + 16} \Rightarrow$$

Translation in  $s$  property

$$(s^2 - 5s + 4)Y(s) = \frac{4}{(s+3)^2 + 16} - 1 \Rightarrow$$

$$Y(s) = \frac{1}{s^2 - 5s + 4} \left( \frac{4}{(s+3)^2 + 16} - 1 \right) = \frac{-(s+3)^2 - 12}{(s^2 - 5s + 4)((s+3)^2 + 16)} = \frac{-s^2 + 6s + 21}{(s^2 - 5s + 4)((s+3)^2 + 16)}$$

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$$b) \quad y'' + 4y' + 5y = t^4, \quad y(0) = 1, \quad y'(0) = 0$$

Apply Laplace transform:

$$\begin{array}{l|l} 5 \times & L\{y\} = Y(s) \\ + 4 \times & L\{y'\} = sY(s) - y(0) = sY(s) - 1 \\ + 1 \times & L\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s \end{array}$$

$$L\{y'' + 4y' + 5y\} = (s^2 + 4s + 5)Y(s) - s - 1 = L\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5} \Rightarrow$$

$$Y(s) = \boxed{\frac{1}{s^2 + 4s + 5} \left( \frac{24}{s^5} + s + 1 \right)} = \frac{1}{s^2 + 4s + 5} \cdot \frac{s^6 + 4s^5 + 24}{s^5}$$