Homework assignment 12 solutions, MATH 308

Problem 1

(a) $$y''-6y'+13y=\underbrace{3t^4e^{3t}}_{g_1(t)}\underbrace{-(t^3+1)e^{2t}\sin 3t}_{g_2(t)}+\underbrace{(t^2-1)e^{2t}}_{g_3(t)}\underbrace{-t^2e^{3t}\cos 2t}_{g_4(t)} \quad (1)$$

Roots of the characteristic equation: $r^2-6r+13=0$

$D=36-52=-16$

$r_{1,2}=\frac{6\pm 4i}{2}=3\pm 2i$

Consider separately 4 equations with the same left-hand side and right-hand sides g_1, g_2, g_3, and g_4

- $y''-6y'+13y=3t^4e^{3t}$ (1a)

  $\alpha=3$ is <u>not</u> a root of the characteristic equation $\Rightarrow s=0$

  Since the degree of the polynomial $n=4$, a solution can be found in the form

  $$y_1(t)=(A_4t^4+A_3t^3+A_2t^2+A_1t+A_0)e^{3t}$$

- $y''-6y'+13y=-(t^3+1)e^{2t}\sin 3t$ $\rightarrow$ trigonometric case

  $\alpha=2,\ \beta=3 \Rightarrow \alpha+i\beta=2+i3$ is <u>not</u> a root of the characteristic equation $\Rightarrow s=0$ and since the degree of the polynomial $n=3$, a solution can be found in the form

  $$y_2(t)=(B_3t^3+B_2t^2+B_1t+B_0)e^{2t}\cos 3t+(C_3t^3+C_2t^2+C_1t+C_0)e^{2t}\sin 3t$$

- $y''-6y'+13y=(t^2-1)e^{2t}$

  $\alpha=2$ is <u>not</u> a root of the characteristic polynomial $\Rightarrow s=2$ and since the degree of the polynomial $n=2$, a solution can be found in the form $y_3(t)=(D_2t^2+D_1t+D_0)e^{2t}$

- $y'' - 6y' + 13y = -t^2 e^{3t}\cos 2t \to$ trigonometric case

$\alpha = 3,\ \beta = 2 \Rightarrow \alpha + i\beta = 3+2i$. It is a root of characteristic polynomial and it appears once among the roots $\Rightarrow s = 1$. Also the degree of the polynomial $n = 2 \Rightarrow$ a solution can be found in the form $y_4(t) = te^{3t}\cos 2t\,(E_2t^2 + E_1t + E_0) + te^{3t}\sin 2t\,(F_2t^2 + F_1t + F_0)$

Combining all together we have to look for a solution in the form

$$\boxed{\begin{aligned}&(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0)e^{3t} + (B_3t^3 + B_2t^2 + B_1t + B_0)e^{2t}\cos 3t + \\ &+ (C_3t^3 + C_2t^2 + C_1t + C_0)e^{2t}\sin 3t + (D_2t^2 + D_1t + D_0)e^{2t} + \\ &+ (E_2t^2 + E_1t + E_0)te^{3t}\cos 2t + (F_2t^2 + F_1t + F_0)te^{3t}\sin 2t\end{aligned}}$$

16) $y'' - 10y' + 21y = t^3e^{3t}\sin 7t - 5te^{7t}\cos 3t + 5t^3e^{3t} - 4te^{3t} + t^2e^{5t}$

Roots of characteristic equation

$r^2 - 10r + 21 = 0 \Rightarrow D = 100 - 84 = 16 \to r_1 = \frac{10+4}{2} = 7,\ r_2 = \frac{10-4}{2} = 3$

Again consider several equations separately according to the number of terms in the right-hand side of the original equation.

- $y'' - 10y' + 21y = t^3e^{3t}\sin 7t \to$ trigonometric case

$\alpha = 3,\ \beta = 7$. $\alpha + i\beta = 3 + i7$ is not a root of the characteristic equation $\Rightarrow s = 0$. Also the degree of the polynomial $n = 3 \Rightarrow$ one looks for a solution in the form $y_1(t) = (A_3t^3 + A_2t^2 + A_1t + A_0)e^{3t}\cos 7t + (B_3t^3 + B_2t^2 + B_1t + B_0)e^{3t}\sin 7t$

(pg 3) • $y''-10y'+21y = -5t\, e^{7t}\cos 3t$ – trigonometric case

$\alpha=7,\ \beta=3$. $\alpha+i\beta = 7+i3$ is <u>not</u> a root of the characteristic equation $\Rightarrow s=0$. Also the degree of the polynomial factor is $n=1 \Rightarrow$ One looks for a solution in the form

$$y_2(t) = (C_1t+C_0)e^{7t}\cos 3t + (D_1t+D_0)e^{7t}\sin 3t$$

• $y''-10y'+21y = (5t^3-4t)e^{3t}$ (here we combined the 3rd and 4th terms)

$\alpha=3$ is a <u>nonrepeated root</u> of the characteristic equation $\Rightarrow s=1$

Also the degree of the polynomial factor is $n=3 \Rightarrow$ one looks for a solution in the form $y_3(t) = (E_3t^3+E_2t^2+E_1t+E_0)te^{3t}$

• $y''-10y'+21y = t^2e^{5t}$

$\alpha=5$ is <u>not</u> a root of the characteristic polynomial $\Rightarrow s=0$

Also the polynomial factor of the right-hand side has degree $2 \Rightarrow$

One looks for a solution in the form

$$y_4(t) = (F_2t^2+F_1t+F_0)e^{5t}$$

Combining all we get

$$\boxed{\begin{aligned} y(t) =\ & (A_3t^3+A_2t^2+A_1t+A_0)e^{3t}\cos 7t + (B_3t^3+B_2t^2+B_1t+B_0)\times \\ & e^{3t}\sin 7t + (C_1t+C_0)e^{7t}\cos 3t + (D_1t+D_0)e^{7t}\sin 3t + (E_3t^3+E_2t^2 \\ & +E_1t+E_0)te^{3t} + (F_2t^2+F_1t+F_0)e^{5t} \end{aligned}}$$

(c) $y''+16y'+64y = te^{-8t}\cos 10t + te^{-8t} - t^3e^{8t}$

The roots of the characteristic polynomials: $r^2+16r+64=0 \Leftrightarrow$

$(r+8)^2=0 \Rightarrow r_{1,2}=-8$

Consider 3 separate equation according to the 3 terms in the right hand side:

- $y''+16y'+64y = te^{-8t}\cos 10t$ → trigonometric case

  $\alpha=-8, \beta=10 \Rightarrow \alpha+i\beta=-8+10i$ is not a root of the characteristic polynomial $\Rightarrow s=0$. Also the degree of the polynomial factor of the right-hand side is $1 \Rightarrow$ one looks for a solution in the form: $y_1(t) = (A_1t+A_0)e^{-8t}\cos 10t + (B_1t+B_0)e^{-8t}\sin 10t$

- $y''+16y'+64y = te^{-8t}$

  $\alpha=-8$ is a repeated root of the characteristic equation $\Rightarrow s=2$

  Also the degree of the polynomial part of the right-hand side is equal to 1.

  Therefore we look for a solution in the form

  $y_2(t) = (C_1t+C_2)t^2e^{-8t}$

- $y''+16y'+64y = -t^3e^{8t}$

  $\alpha=8$ is not a root of the characteristic polynomial, so $s=0$

  Also the degree of the polynomial factor of the right-hand side is $3 \Rightarrow$

  we look for a solution in the form

  $y_3(t) = (D_3t^3+D_2t^2+D_1t+D_0)e^{8t}$

Combining all together we get that a solution of the original equation can be found in the form

$$\boxed{\begin{aligned} y(t) = {} & (A_1t+A_0)e^{-8t}\cos 10t + (B_1t+B_0)e^{-8t}\sin 10t + \\ & +(C_1t+C_2)t^2e^{-8t} + (D_3t^3+D_2t^2+D_1t+D_0)e^{8t} \end{aligned}}$$

Problem 2 $\quad W = 4lb, \quad L = 2in = \frac{2}{12} ft = \frac{1}{6} ft, \quad \gamma = 4,$

$$k = \frac{W}{L} = \frac{4}{\frac{1}{6}} = 24 \qquad F(t) = 2\cos 4t + 4\sin 4t$$

$$m = \frac{W}{g} = \frac{4}{32} = \frac{1}{8}$$

The equation is $mu'' + \gamma u' + ku = F(t) \iff$

$$\frac{1}{8}u'' + 4u' + 24u = 2\cos 4t + 4\sin 4t$$

$\alpha = 0, \beta = 4i$. $\alpha + i\beta = 4i$ cannot be a root of the characteristic equation, because the roots of the characteristic equation have a nonzero real part $\Rightarrow s = 0$. Also the polynomial factors of the right-hand side are constant $\Rightarrow n = 0$

We look for a solution in the form

$$u(t) = A\cos 4t + B\sin 4t$$

$$u'(t) = 4B\cos 4t - 4A\sin 4t$$

$$u''(t) = -16A\cos 4t - 16B\sin 4t$$

$\Rightarrow \frac{1}{8}u'' + 4u' + 24u = (-2A + 16B + 24A)\cos 4t + (-2B - 16A + 24B)\sin 4t =$

$= (22A + 16B)\cos 4t + (-16A + 22B)\sin 4t = 2\cos 4t + 4\sin 4t$

$$\Rightarrow \begin{cases} 22A + 16B = 2 \\ -16A + 22B = 4 \end{cases} \iff \begin{cases} 11A + 8B = 1 & \times 11 \\ -8A + 11B = 2 & \times 8 \end{cases}$$

11(Eq 1) − 8 Eq 2 $\Rightarrow$ $\underbrace{(121 + 64)}_{185}A = \underbrace{11 - 16}_{-5} \Rightarrow$

$$\Rightarrow A = -\frac{5}{185} = -\frac{1}{37} \Rightarrow B = \frac{1}{8}\left(1 + \frac{11}{37}\right) = \frac{48}{8 \cdot 37} = \frac{6}{37} \Rightarrow$$

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$\Rightarrow \boxed{u(t) = -\frac{1}{37}\cos 4t + \frac{6}{37}\sin 4t}$

Problem 3

(a) $\mathcal{L}^{-1}\left\{\frac{s^2-1}{(s^2-4s+4)(s+4)}\right\}$

$s^2-4s+4=(s-2)^2 \Rightarrow$

We look for the partial fraction decomposition in the form

$$\frac{s^2-1}{(s-2)^2(s+4)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+4} \Rightarrow$$

$$s^2-1 = A(s-2)(s+4) + B(s+4) + C(s-2)^2$$

To find B plug $s=2$: $\underbrace{4-1}_{3} = 6B \Rightarrow B = \frac{1}{2}$

To find C plug $s=-4$: $(-4)^2-1 = C(-4-2)^2 \Leftrightarrow 15 = 36C \Rightarrow C = \frac{5}{12}$

To find A compare coefficients of s^2:

$$1 = A + C \Rightarrow A = 1 - C = 1 - \frac{5}{12} = \frac{7}{12}$$

$$\Rightarrow \frac{s^2-1}{(s-2)^2(s+4)} = \frac{7}{12}\frac{1}{s-2} + \frac{1}{2}\frac{1}{(s-2)^2} + \frac{5}{12}\frac{1}{s+4} \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2-1}{(s-2)^2(s+4)}\right\} = \frac{7}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + \frac{5}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} \underset{\text{using the table}}{=}$$

$$= \boxed{\frac{7}{12}e^{2t} + \frac{1}{2}te^{2t} + \frac{5}{12}e^{-4t}}$$

(b) $\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-5)(9s^2+18s+90)}\right\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-5)(s^2+2s+10)}\right\}$

$$s^2+2s+10 = (s+1)^2+9 \rightarrow \alpha=-1,\ \beta=3$$

We look for the partial fraction decomposition in the form

$$\frac{4s+5}{(s-5)((s+1)^2+9)} = \frac{A}{s-5} + \frac{B(s+1)+3C}{(s+1)^2+9} \iff$$

$$4s+5 = A\left((s+1)^2+9\right) + (s-5)\left(B(s+1)+3C\right)$$

To find A plug $s=5$: $4\cdot 5+5 = A\left((5+1)^2+9\right) \iff 25 = 45A \Rightarrow$

$A = \frac{5}{9}$

To find C plug $s=-1$: $4\cdot(-1)+5 = 9\underset{5/9}{A} + (-1-5)\cdot 3C \iff$

$$1 = 5 - 18C \Rightarrow C = \frac{2}{9}$$

To find B compare coefficients of s^2:

$$0 = A + B \Rightarrow B = -A = -\frac{5}{9}$$

So $\frac{4s+5}{(s-5)((s+1)^2+9)} = \frac{5}{9}\frac{1}{s-5} - \frac{5}{9}\frac{s+1}{(s+1)^2+9} + \frac{2}{9}\cdot\frac{3}{(s+1)^2+9} \Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-5)((s+1)^2+9)}\right\} = \frac{5}{9}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}}_{e^{5t}} - \frac{5}{9}\underbrace{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}}_{e^{-t}\cos 3t} + \frac{2}{9}\underbrace{\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}}_{e^{-t}\sin 3t}$$

$$= \frac{1}{9}\left(5e^{5t} - 5e^{-t}\cos 3t + 2e^{-t}\sin 3t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-5)(9s^2+18s+90)}\right\} = \frac{1}{9} \text{ of the previous } = \boxed{\frac{1}{81}\left(5e^{5t} - 5e^{-t}\cos 3t + 2e^{-t}\sin 3t\right)}$$

c) $\mathcal{L}^{-1}\left\{\frac{s^3}{(s^2+9)^2}\right\}$

$s^2+9=(s-3i)(s+3i)$ => the complex-valued partial fraction decomposition

$$\frac{s^3}{(s^2+9)^2}=\frac{s^3}{(s-3i)^2(s+3i)^2}=\frac{A}{s-3i}+\frac{B}{(s-3i)^2}+\frac{C}{(s+3i)}+\frac{D}{(s+3i)^2} \quad (\Leftarrow)$$

$$s^3=A(s-3i)(s+3i)^2+B(s+3i)^2+C(s+3i)(s-3i)^2+D(s-3i)^2$$

To find B plug $s=3i$: $(3i)^3=B(2\cdot 3i)^2 \Leftrightarrow 3i=4B \Rightarrow \boxed{B=\frac{3}{4}i}$

To find D plug $s=-3i$: $(-3i)^3=D((-2)\cdot 3i)^2 \Rightarrow -3i=4D \Rightarrow D=-\frac{3}{4}i$

For A & C 1) compare coefficients of s^3:

$$1=A+C$$

2) plug $s=0$: $0=-A\cdot(3i)^3+B\cdot(3i)^2+C(3i)^3+D(3i)^2=$

$i^3=-i \longrightarrow$ $= 27i(C-A)+\underbrace{(B+D)}_{0}\cdot(-9)=27i(C-A) \Rightarrow C-A=0$
$A=C$

$$\begin{cases} A+C=1 \\ A=C \end{cases} \Rightarrow \boxed{A=C=\frac{1}{2}}$$

$$\Rightarrow \frac{s^3}{(s^2+9)^2}=\frac{1}{2}\frac{1}{s-3i}+\frac{3}{4}i\frac{1}{(s-3i)^2}+\frac{1}{2}\frac{1}{s+3i}-\frac{3}{4}i\frac{1}{(s-3i)^2} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^2+9}\right\}=\frac{1}{2}e^{3it}+\frac{3}{4}ite^{3it}+\frac{1}{2}e^{-3it}-\frac{3}{4}ite^{-3it}=$$

$$= \cos 3t - \frac{3}{2} t \underbrace{\frac{e^{3it} - e^{-3it}}{2i}}_{\sin 3t} = \boxed{\cos 3t - \frac{3}{2} t \sin 3t}$$

Problem 4

$$y'' - 2y' + 10y = 5\sin t \quad , y(0) = -1 , y'(0) = 1$$

$$10 \times \quad \mathcal{L}\{y\} = Y(s)$$

$$+ \quad -2 \times \quad \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 1$$

$$+ \quad \mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) + s - 1$$

$$\mathcal{L}\{y'' - 2y' + 10y\} = (s^2 - 2s + 10s) Y(s) + s - 1 - 2 =$$

$$= (s^2 - 2s + 10s) Y(s) + s - 3 = \mathcal{L}\{5\sin t\} = 5 \frac{1}{s^2+1} \Rightarrow$$

$$Y(s) = \frac{1}{s^2 - 2s + 10} \left(\frac{5}{s^2+1} - s + 3 \right) = \frac{1}{s^2 - 2s + 10} \left(\frac{5 - s^3 + 3s^2 - s + 3}{s^2+1} \right) =$$

$$= \frac{-s^3 + 3s^2 - s + 8}{(s^2 - 2s + 10)(s^2+1)} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$s^2 - 2s + 10 = (s-1)^2 + 9 \Rightarrow \alpha = 1, \beta = 3 \Rightarrow$ the partial fraction decomposition is of the form

$$\frac{-s^3 + 3s^2 - s + 8}{((s-1)^2 + 9)(s^2+1)} = \frac{A(s-1) + 3B}{(s-1)^2 + 9} + \frac{Cs + D}{s^2+1} \Rightarrow$$

$$-s^3 + 3s^2 - s + 8 = (\underbrace{A(s-1) + 3B}_{As + 3B - A})(s^2+1) + (Cs + D)(\underbrace{(s-1)^2 + 9}_{s^2 - 2s + 10}) =$$

$$As^3 + (3B - A)s^2 + As + 3B - A + Cs^3 + Ds^2 - 2Cs^2 - 2Ds + 10Cs + 10D$$

$$= (A+C)s^3 + (3B - A + D - 2C)s^2 + (A + 10C - 2D)s + 3B - A + 10D$$

$\Rightarrow$ comparing coefficients:

$$\begin{cases} A + C = -1 \Rightarrow A = -1 - C \\ -A + 3B - 2C + D = 3 \\ A + 10C - 2D = -1 \\ -A + 3B + 10D = 8 \end{cases} \qquad \begin{cases} 1 + C + 3B - 2C + D = 3 \\ \cancel{-1} - C + 10C - 2D = \cancel{-1} \\ 1 + C + 3B + 10D = 8 \end{cases} \Leftrightarrow$$

$$\begin{cases} 3B - C + D = 2 \\ 9C - 2D = 0 \\ 3B + C + 10D = 7 \end{cases}$$

Augmented matrix (actually we could also write 4×5 augmented matrix for the original system of 4 equations)

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 0 & 9 & -2 & 0 \\ 3 & 1 & 10 & 7 \end{array}\right) \underset{\sim}{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 0 & 9 & -2 & 0 \\ 0 & 2 & 9 & 5 \end{array}\right)$$

$$\underset{\sim}{R_2 \to 9R_3 - 2R_2} \left(\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 0 & 9 & -2 & 0 \\ 0 & 0 & 85 & 45 \end{array}\right) \Rightarrow$$

$3B - C + D = 2 \qquad 3B - \frac{2}{17} + \frac{9}{17} = 2 \Rightarrow$

$\Rightarrow B = \frac{1}{3}\left(\frac{34-7}{17}\right) = \boxed{\frac{9}{17}}$

$9C - 2D = 0 \Rightarrow C = \frac{2}{9}D = \boxed{\frac{2}{17}}$

$85D = 45 \Rightarrow D = \frac{45}{85} = \boxed{\frac{9}{17}}$

Also $A = -1 - C = -1 - \frac{2}{17} = -\frac{19}{17} \Rightarrow$

$$y(t) = \underset{-\frac{19}{17}}{A}\, \underbrace{\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+9}\right\}}_{e^t \cos 3t} + \underset{\frac{9}{17}}{B}\, \underbrace{\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2+9}\right\}}_{e^t \sin 3t} + \underset{\frac{2}{17}}{C}\, \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos t} + \underset{\frac{9}{17}}{D}\, \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin t}$$

$$= \boxed{-\frac{19}{17} e^t \cos 3t + \frac{9}{17} e^t \sin 3t + \frac{2}{17} \cos t + \frac{9}{17} \sin t}$$