Homework #12 solutions - MATH 308 - Summer 2012

Problem 1

a) $s^2+4s-12 = (s+6)(s-2) \Rightarrow$

$$F(s) = \frac{3s-4}{(s+6)(s-2)^2} \Rightarrow$$

The partial fraction decomposition is

$$\frac{3s-4}{(s+6)(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+6} \Rightarrow$$

$$3s-4 = A(s-2)(s+6) + B(s+6) + C(s-2)^2 \quad (*)$$

To find B put $s=2$ into $(*)$

$$6-4 = B\cdot 8 \Rightarrow B = \frac{2}{8} = \frac{1}{4}$$

To find C put $s=-6$ into $(*)$

$$-18-4 = C(-6-2)^2 \Rightarrow -22 = 64C \Rightarrow C = \frac{-22}{64} = -\frac{11}{32}$$

To find A compare coefficients of s^2:

$$0 = A + C \Rightarrow A = -C = \frac{11}{32} \Rightarrow$$

$$F(s) = \frac{3s-4}{(s+6)(s-2)^2} = \frac{11}{32}\frac{1}{s-2} + \frac{1}{4}\frac{1}{(s-2)^2} - \frac{11}{32}\frac{1}{s+6} \Rightarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = \boxed{\frac{11}{32}e^{2t} + \frac{1}{4}te^{2t} - \frac{11}{32}e^{-6t}}$$

Problem 1

b) $s^2+2s+17 = (s+1)^2+16 \Rightarrow \alpha=-1, \beta=4 \Rightarrow$

We look for the partial fraction decomposition in the form

$$F(s) = \frac{-5s+1}{((s+1)^2+16)(s+1)} = \frac{A(s+1)+B\cdot 4}{(s+1)^2+16} + \frac{C}{s+1} \Rightarrow$$

$$-5s+1 = (A(s+1)+B\cdot 4)(s+1) + C((s+1)^2+16)$$

To find C let $s=-1 \Rightarrow$

$$5+1 = C\cdot 16 \Rightarrow C = \frac{6}{16} = \frac{3}{8}$$

To find A compare coefficients of s^2:

$$0 = A + C \Rightarrow A = -C = -\frac{3}{8}$$

To find B let $s=0$:

$$1 = A + 4B + 17C = -\frac{3}{8} + 4B + \frac{51}{8} \Rightarrow 4B = 1 - \frac{48}{8} = 1-6 = -5$$

$$B = -\frac{5}{4} \Rightarrow$$

$$F(s) = -\frac{3}{8}\frac{s+1}{(s+1)^2+4^2} - \frac{5}{4}\frac{4}{(s+1)^2+4^2} + \frac{3}{8}\frac{1}{s+1} \Rightarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = \boxed{-\frac{3}{8}e^{-t}\cos 4t - \frac{5}{4}e^{-t}\sin 4t + \frac{3}{8}e^{-t}}$$

Problem 2

$-15\times \ \mathcal{L}\{y\} = Y$

$2\times \ \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 1$

$1\times \ \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 2$

$$\mathcal{L}\{y''+2y'-15y\} = (s^2+2s-15)Y(s) + s-2+2 = 10\,\mathcal{L}\{e^{3t}\sin 2t\} = 10\cdot\frac{2}{(s-3)^2+4} = \frac{20}{(s-3)^2+4} \Rightarrow$$

$$(s^2+2s-15)Y(s) = -s + \frac{20}{(s-3)^2+4} \Rightarrow Y(s) = \frac{-s}{s^2+2s-15} + \frac{20}{(s^2+2s-15)((s-3)^2+4)}$$

$s^2+2s-15 = (s+5)(s-3)$

i. Find the partial fraction decomposition of $\frac{s}{s^2+2s-15} = \frac{s}{(s+5)(s-3)}$

$$\frac{s}{(s+5)(s-3)} = \frac{A}{s+5} + \frac{B}{s-3}$$

$$s = A(s-3) + B(s+5)$$

$s=3 \Rightarrow \quad 3 = B\cdot 8 \Rightarrow B = \frac{3}{8}$

$s=-5 \Rightarrow \quad -5 = A(-5-3) \Rightarrow \quad 5 = 8A \Rightarrow A = \frac{5}{8} \Rightarrow$

$$\frac{s}{(s+5)(s-3)} = \frac{5}{8}\frac{1}{s+5} + \frac{3}{8}\frac{1}{s-3} \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s}{(s+5)(s-3)}\right\} = \frac{5}{8}e^{-5t} + \frac{3}{8}e^{3t}$$

ii. Find the partial fraction decomposition of $\frac{20}{(s+5)(s-3)((s-3)^2+4)}$

$$\frac{20}{(s+5)(s-3)((s-3)^2+4)} = \frac{A}{s+5} + \frac{B}{s-3} + \frac{C(s-3)+D\cdot 2}{(s-3)^2+4}$$

$$20 = A(s-3)((s-3)^2+4) + B(s+5)((s-3)^2+4) + (C(s-3)+D\cdot 2)(s+5)(s-3)$$

$s=-5 \Rightarrow \quad 20 = A(-8)(8^2+4) = -8\cdot 68\, A \Rightarrow \boxed{A = -\frac{20}{8\cdot 68} = -\frac{5}{136}}$

$s=3 \Rightarrow \quad 20 = B\cdot 8\cdot 4 = 32B \Rightarrow \boxed{B = \frac{20}{32} = \frac{5}{8}}$

Compare coefficient of s^3 (i.e. of the leading term):

$$0 = A+B+C \Rightarrow C = -A-B = +\frac{5}{136} - \frac{5}{8} = -\frac{5}{8}\left(-\frac{1}{17}+1\right) =$$

$$= -\frac{5}{8}\cdot\frac{16}{17} = -\frac{5\cdot 2}{17} = -\frac{10}{17} \Rightarrow \boxed{C = -\frac{10}{17}}$$

$s=0 \Rightarrow \quad 20 = A\cdot(-3)\cdot 13 + B\cdot 5\cdot 13 + (-3C+2D)(-15) =$

$$= -39A + 65B + 45C - 30D = 39\,\frac{5}{136} + 65\cdot\frac{5}{8} - \frac{450}{17} - 30D \Rightarrow$$

$$D = \frac{1}{30}\left(\frac{195}{136} + \frac{325}{8} - \frac{450}{17} - 20\right) = \frac{1}{30}\,\frac{195+325\cdot 17 - 450\cdot 8 - 20\cdot 136}{136} =$$

$$= \frac{1}{30}\left(\frac{195+5525-3600-2720}{136}\right) = -\frac{1}{30}\frac{600}{136} = -\frac{20}{136} = -\frac{5}{34} \Rightarrow \boxed{D = -\frac{5}{34}}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{(s+5)(s-3)((s-3)^2+4)}\right\} = -\frac{5}{136}e^{-5t} + \frac{5}{8}e^{3t} - \frac{10}{17}e^{3t}\cos 2t - \frac{5}{34}e^{3t}\sin 2t \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{5}{8}e^{-5t} - \frac{3}{8}e^{3t} - \frac{5}{136}e^{-5t} + \frac{5}{8}e^{3t} - \frac{10}{17}e^{3t}\cos 2t - \frac{5}{34}e^{3t}\sin 2t = \boxed{-\frac{45}{68}e^{-5t} + \frac{1}{4}e^{3t} - \frac{10}{17}e^{3t}\cos 2t - \frac{5}{34}e^{3t}\sin 2t}$$

<u>Problem 3</u> $f(t) = 2 + (t-2-2)\,u_2(t) + ((1-2t)-(t-2))\,u_3(t) =$

$$= 2 + (t-4)\,u_2(t) + (3-3t)\,u_3(t) = 2 + (t-4)\,u_2(t) - 3(t-1)\,u_3(t)$$

We use here the translation in t property:

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}F(s)$$

i) $\mathcal{L}\{2\} = \frac{2}{s}$

ii) $\mathcal{L}\{(t-4)\,u_2(t)\}$?

<u>Way one</u> Find $f_1(t)$ s.t. $f_1(t-2) = t-4 \Rightarrow f_1(t) = f_1((t+2)-2) =$

$$= t+2-4 = t-2 \Rightarrow F_1(s) = \mathcal{L}\{f_1(t+2)\} = \mathcal{L}\{t-2\} = \frac{1}{s^2} - \frac{2}{s} \Rightarrow$$

$$\mathcal{L}\{(t-4)\,u_2(t)\} = e^{-2s}F_1(s) = e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) = e^{-2s}\,\frac{1-2s}{s^2}$$

<u>Way two</u> $(t-4)\,u_2(t) = (t-2)\,u_2(t) - 2u_2(t) \Rightarrow$

$$\mathcal{L}\{(t-4)\,u_2(t)\} = e^{-2s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s} = e^{-2s}\,\frac{1-2s}{s^2}$$

iii) $\mathcal{L}\{(t-1)\,u_3(t)\}$:

<u>Way one</u> Find $f_2(t)$ s.t $f_2(t-3) = t-1 \Rightarrow f_2(t) = f_2((t+3)-3) =$

$$= t+3-1 = t+2 \Rightarrow F_2(s) = \mathcal{L}\{f_2(t)\} = \frac{1}{s^2} + \frac{2}{s} = \frac{2s+1}{s^2}$$

$$\mathcal{L}\{(t-1)\,u_3(t)\} = e^{-3s}F_2(s) = e^{-3s}\,\frac{2s+1}{s^2}$$

<u>Way two</u> $(t-1)\,u_3(t) = (t-3+2)\,u_3(t) = (t-3)\,u_3(t) + 2u_3(t) \Rightarrow$

$$\mathcal{L}\{(t-1)\,u_3(t)\} = \mathcal{L}\{(t-3)\,u_3(t)\} + 2\mathcal{L}\{u_3(t)\} = e^{-3s}\frac{1}{s^2} + 2e^{-3s}\frac{1}{s} =$$

$$= e^{-3s}\,\frac{2s+1}{s^2},$$

Combining (i), (ii), (iii) we get

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s} + e^{-2s}\,\frac{1-2s}{s^2} - 3e^{-3s}\,\frac{2s+1}{s^2}}$$