Homework assignment # 14 MATH 308 FALL 2012

Problem 1

$$\begin{cases} x_1' = -3x_1 + 2x_2 \\ x_2' = -3x_1 + 4x_2 \end{cases}$$

(a) Find the general solution

$$A = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}$$

i) Char eq. is $\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0$; $\mathrm{tr}A = 1$, $\det A = -12+6 = -6$

$$\lambda^2 - \lambda - 6 = 0$$
$$D = 1+24 = 25$$
$$\lambda_1 = \frac{1+5}{2} = 3$$
$$\lambda_2 = \frac{1-5}{2} = -2$$

ii) Find ein eigenvector of $\lambda_1 = 3$

$$A - 3I = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow -3v_1 + v_2 = 0 \Rightarrow v_2 = 3v_1$$

Fix $v_1 = 1 \Rightarrow v_2 = 3 \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\lambda_1 = 3 \Rightarrow$

$e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is a solution

iii) Find an eigenvector of $\lambda_2 = -2$

$$A - (-2)I = A + 2I = \begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$-v_1 + 2v_2 = 0 \Rightarrow v_1 = 2v_2$$

Fix $v_2 = 1 \Rightarrow v_1 = 2 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda_2 = -2 \Rightarrow$

$e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a solution

$\Downarrow$

The general solution is

$$x(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Find the solution of the system (1) satisfying the initial conditions $x_1(0)=2, x_2(0)=-1$

$$C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{matrix} C_1 + 2C_2 = 2 \\ 3C_1 + C_2 = -1 \end{matrix} \Rightarrow \text{Eq } 2 - 3\text{Eq } 1 \Rightarrow -5C_2 = -7 \Rightarrow C_2 = \frac{7}{5}$$

$$C_1 = 2 - 2C_2 = 2 - \frac{14}{5} = -\frac{4}{5}$$

$\Downarrow$

$$x(t) = -\frac{4}{5}e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{7}{5}e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(c) $x(t) \xrightarrow[t\to\infty]{} 0 \iff C_1 = 0 \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(d) $x(t) \xrightarrow[t\to-\infty]{} 0 \iff C_2 = 0 \Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Problem 2

$$\begin{cases} x_1' = 2x_1 + 3x_2 - 3x_3 \\ x_2' = x_1 + 2x_2 - x_3 \\ x_3' = x_1 + 3x_2 - 2x_3 \end{cases}$$

$$A = \begin{pmatrix} 2 & 3 & -3 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$$

(a) i) Find the eigenvalues of A

$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 3 & -3 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{pmatrix} = (2-\lambda)\big(\underbrace{(2-\lambda)(-2-\lambda)}_{(\lambda-2)(\lambda+2)} + 3\big) -$$

$$-3\,(\underbrace{-2-\lambda+1}_{-\lambda-1}) - 3\,(\underbrace{3-2+\lambda}_{1+\lambda}) = (2-\lambda)(\lambda^2 - 4 + 3)$$

$$= (2-\lambda)(\lambda-1)(\lambda+1) + 3\cancel{(\lambda+1)} - 3\cancel{(\lambda+1)} = -(\lambda-2)(\lambda-1)(\lambda+1) =$$

$$0 \Rightarrow \quad \lambda_1 = 1, \ \lambda_2 = -1, \ \lambda_3 = 2$$

ii) Find an eigenvector corresponding to $\lambda_1 = 1$

$$A - \lambda_1 I = A - I = \begin{pmatrix} 1 & 3 & -3 \\ 1 & 1 & -1 \\ 1 & 3 & -3 \end{pmatrix}$$

Solve $(A - I)v = 0$ using the Gauss elimination method

Augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$v_1 + 3v_2 - 3v_3 = 0$$
$$-2v_2 + 2v_3 = 0$$

Fix $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow v_1 + 3 - 3 = 0 \Rightarrow v_1 = 0 \Rightarrow$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresp to $\lambda_1 = 1 \Rightarrow$

$e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a solution of the system (2)

iii) Find an eigenvector corresponding to $\lambda_2 = -1$

$$A - \lambda_2 I = A + I = \begin{pmatrix} 3 & 3 & -3 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \end{pmatrix}$$

Solve $(A+I)v = 0$

$$\left(\begin{array}{ccc|c} 3 & 3 & -3 & 0 \\ 1 & 3 & -1 & 0 \\ 1 & 3 & -1 & 0 \end{array}\right) \underset{R_3 \to R_3 - R_2}{\overset{R_1 \to \frac{1}{3}R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$\left.\begin{array}{l} v_1 + v_2 - v_3 = 0 \\ v_2 = 0 \end{array}\right\} \Rightarrow v_1 - v_3 = 0 \quad \text{Fix } v_3 = 1 \Rightarrow v_1 = 1$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1 \Rightarrow$

$e^{-t}\begin{pmatrix}1\\0\\1\end{pmatrix}$ is a solution of the system (2)

(iv) Find an eigenvector corresponding to $\lambda_3 = 2$

$$A-\lambda_3 I = A-2I = \begin{pmatrix}0 & 3 & -3\\ 1 & 0 & -1\\ 1 & 3 & -4\end{pmatrix}$$

Solve $(A-2I)v=0$

$$\left(\begin{array}{ccc|c}0 & 3 & -3 & 0\\ 1 & 0 & -1 & 0\\ 1 & 3 & -4 & 0\end{array}\right) \overset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 3 & -3 & 0\\ 1 & 3 & -4 & 0\end{array}\right) \overset{R_3 \to R_3 - R_1}{\sim} \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 3 & -3 & 0\\ 0 & 3 & -3 & 0\end{array}\right) \overset{R_3 \to R_3 - R_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 3 & -3 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \overset{R_2 \to \frac{1}{3}R_2}{\sim} \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \Rightarrow$$

$$v_1 - v_3 = 0$$
$$v_2 - v_3 = 0$$

Fix $v_3 = 1 \Rightarrow v_2 = 1,\ v_1 = 1 \Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2 \Rightarrow e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a solution

$\Downarrow$ Combining ii), iii), iv)

The general solution is

$$x(t) = C_1 e^{t}\begin{pmatrix}0\\1\\1\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\0\\1\end{pmatrix} + C_3 e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(b) $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \Rightarrow$

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{array}\right) \underset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right) \underset{R_3 \to R_3 - R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right)$$

$$\underset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{array}\right) \Rightarrow \begin{array}{ll} c_1 + c_3 = -1 & c_1 - 1 = -1 \Rightarrow c_1 = 0 \\ c_2 + c_3 = 2 & c_2 - 1 = 2 \Rightarrow c_2 = 3 \\ -c_3 = 1 & \Rightarrow c_3 = -1 \end{array}$$

$\Downarrow$

$$x(t) = 3e^{-t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$