Homework #13 Solutions - MATH308-SUMMER 2012

Problem 1

First find $\mathcal{L}^{-1}\left\{\frac{2s+2}{(s-2)(s^2-4s+8)}\right\}$

$$s^2-4s+8 = s^2-4s+4+4=(s-2)^2+4 \Rightarrow \alpha=2, \beta=2$$

We look for the following partial fraction decomposition

$$\frac{2s+2}{(s-2)((s-2)^2+4)} = \frac{A(s-2)+B\cdot 2}{(s-2)^2+4} + \frac{C}{s-2}$$

$$2s+2 = \left(A(s-2)+2B\right)(s-2) + C\left((s-2)^2+4\right)$$

To find C put $s=2$:

$$2\cdot 2+2 = 4C \Rightarrow 6=4C \Rightarrow \boxed{C=\frac{3}{2}}$$

To find A compare the coefficients of s^2:

$$0 = A+C \Rightarrow \boxed{A=-\frac{3}{2}}$$

To find B put $s=0$

$$2 = \left(-2A+2B\right)(-2) + C(2^2+4) = 4A-4B+8C = 4\cdot\left(-\frac{3}{2}\right)-4B+8\cdot\frac{3}{2} =$$

$$= -6-4B+12 = -4B+6 \Rightarrow 2=-4B+6 \Rightarrow 4B=4 \Rightarrow \boxed{B=1}$$

$\Downarrow$

$$\frac{2s+2}{(s-2)((s-2)^2+4)} = -\frac{3}{2}\frac{s-2}{(s-2)^2+4} + \frac{2}{(s-2)^2+4} + \frac{3}{2}\frac{1}{s-2} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{2s+2}{(s-2)((s-2)^2+4)}\right\} = -\frac{3}{2}e^{2t}\cos 2t + e^{2t}\sin 2t + \frac{3}{2}e^{2t} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{2s+2}{(s-2)((s-2)^2+4)}\right\} = \left(-\frac{3}{2}e^{2(t-5)}\cos 2(t-5) + e^{2(t-5)}\sin 2(t-5) + \frac{3}{2}e^{2(t-5)}\right)u_5(t) =$$

$$= \boxed{e^{2t-10}\left(-\frac{3}{2}\cos(2t-10)+\sin(2t-10)+\frac{3}{2}\right)u_5(t)}$$

<u>Problem 2</u> (Problem 9(a), page 336 of the textbook)

$$y''+y=g(t),\quad y(0)=0,\quad y'(0)=1;\quad g(t)=\begin{cases} t/2, & 0\le t<6 \\ 3, & t\ge 6\end{cases}$$

1) $\mathcal{L}\{g(t)\}$?

$$g(t)=\frac{t}{2}+\left(3-\frac{t}{2}\right)u_6(t)=\frac{t}{2}-\frac{1}{2}(t-6)u_6(t) \Rightarrow$$

$$\mathcal{L}\{g(t)\}=\frac{1}{2}\frac{1}{s^2}-\frac{1}{2}e^{-6s}\frac{1}{s^2}=\frac{1}{2s^2}(1-e^{-6s})$$

$$\mathcal{L}\{y''\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-1$$

$$\mathcal{L}\{y''+y\}=(s^2+1)Y(s)-1=\frac{1}{2s^2}-\frac{e^{-6s}}{2s^2}\Rightarrow$$

$$Y(s)=\frac{1}{s^2+1}+\frac{1}{2s^2(s^2+1)}-\frac{e^{-6s}}{2s^2(s^2+1)}$$

- $\mathcal{L}^{-1}\{\frac{1}{s^2+1}\}=\sin t$
- $\frac{1}{s^2(s^2+1)}=\frac{1}{x(x+1)}$, where $x=s^2$. Find the partial fraction decomposition of $\frac{1}{x(x+1)}$:

$$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}\Rightarrow\quad 1=A(x+1)+Bx\Rightarrow$$

If $x=0$: $1=A$

If $x=-1$: $1=-B\Rightarrow B=-1\Rightarrow$

$$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}\Rightarrow\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}\Rightarrow\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}=t-\sin t$$

$$y(t)=\mathcal{L}^{-1}\{Y(t)\}=\sin t+\frac{1}{2}(t-\sin t)-\frac{1}{2}u_6(t)\left(t-6-\sin(t-6)\right)=$$

$$=\frac{1}{2}t+\frac{1}{2}\sin t-\frac{1}{2}u_6(t)\left(t-\sin(t-6)-6\right)$$

Problem 3 (Problem 3a, page 343 of the textbook)

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$$
$$y(0) = 0$$
$$y'(0) = \frac{1}{2}$$

1) Apply Laplace transform to the right hand side

$$\mathcal{L}\{\delta(t-5)\} = e^{-5s}, \quad \mathcal{L}\{u_{10}(t)\} = \frac{e^{-10s}}{s}$$

2) Apply Laplace transform to the left hand side:

$$\begin{array}{r|l} 2\times & \mathcal{L}\{y\} = Y(s) \\ 3\times & \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) \\ 1\times & \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - \frac{1}{2} \end{array}$$

$$\mathcal{L}\{y'' + 3y' + 2y\} = (s^2+3s+2)Y(s) - \frac{1}{2} = e^{-5s} + \frac{e^{-10s}}{s} \Rightarrow$$

$$Y(s) = \frac{1}{2}\,\frac{1}{s^2+3s+2} + \frac{e^{-5s}}{s^2+3s+2} + \frac{e^{-10s}}{s(s^2+3s+2)}$$

$$s^2+3s+2 = (s+1)(s+2)$$

- $\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + B(s+1) \Rightarrow$ $s=-1$: $1 = A$; $s=-2$: $1 = -B \Rightarrow B = -1 \Rightarrow$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-t} - e^{-2t} \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+3s+2}\right) = u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right)$$

- $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \Rightarrow 1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

To find A put $s=0$: $1 = A\cdot 2 \Rightarrow \boxed{A = \frac{1}{2}}$

To find B put $s=-1$: $1 = B\cdot(-1)\cdot 1 \Rightarrow \boxed{B = -1}$

To find C put $s=-2$: $1 = C(-2)\cdot(-1) \Rightarrow 2C=1 \Rightarrow \boxed{C=\frac{1}{2}} \Rightarrow$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s(s+1)(s+2)}\right\} = u_{10}(t)\left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)$$

$$\Downarrow$$

$$y(t) = \frac{1}{2}(e^{-t} - e^{-2t}) + u_5(t)\left(e^{-t+5} - e^{-2t+10}\right) + u_{10}(t)\left(\frac{1}{2} - e^{-t+10} + \frac{1}{2}e^{-2t+20}\right)$$