Homework assignment #15 _ Solutions, MATH308-505

Problem 1

$$\begin{cases} x_1' = -2x_1 - x_2 \\ x_2' = 16x_1 - 2x_2 \end{cases}$$

(a) General solution $A = \begin{pmatrix} -2 & -1 \\ 16 & -2 \end{pmatrix}$

1) Eigenvalues $\det(A - \lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 + 4\lambda + 20 = (\lambda+2)^2 + 16 = 0 \Rightarrow$

$$\boxed{\begin{array}{l} \operatorname{tr}A = -4 \\ \det A = 4 + 16 = 20 \end{array}}$$

$\lambda_{1,2} = -2 \pm 4i$

2) An eigenvector corresponding to $\lambda = -2 + 4i$

$$\big(A - (-2+4i)I\big)v = \big(A + (2-4i)I\big)v = 0 \Leftrightarrow$$

$$\begin{pmatrix} -4i & -1 \\ 16 & -4i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\left(\begin{array}{cc|c} -4i & -1 & 0 \\ 16 & -4i & 0 \end{array}\right) \xrightarrow{R_2 \to \frac{R_2}{4}} \left(\begin{array}{cc|c} -4i & -1 & 0 \\ 4 & -i & 0 \end{array}\right) \xrightarrow{R_2 \to iR_2 + R_1} \left(\begin{array}{cc|c} -4i & -1 & 0 \\ 0 & \underbrace{1-1}_{0} & 0 \end{array}\right)$$

$\Rightarrow \quad -4i\,v_1 - v_2 = 0 \Rightarrow v_2 = -4i\,v_1$

Set $v_1 = 1 \Rightarrow v_2 = -4i \Rightarrow v = \begin{pmatrix} 1 \\ -4i \end{pmatrix} \Rightarrow$

$\Rightarrow$ the fundamental set of solutions is

$$\mathrm{Re}\left(e^{(-2+4i)t}\begin{pmatrix}1\\-4i\end{pmatrix}\right) \text{ and } \mathrm{Im}\left(e^{(-2+4i)t}\begin{pmatrix}1\\-4i\end{pmatrix}\right)$$

$$e^{(-2+4i)t}\begin{pmatrix}1\\-4i\end{pmatrix} = e^{-2t}(\cos 4t + i\sin 4t)\begin{pmatrix}1\\-4i\end{pmatrix} =$$

$$= e^{-2t}\begin{pmatrix}\cos 4t + i\sin 4t\\ 4\sin 4t - 4i\cos 4t\end{pmatrix} \Rightarrow$$

$$\mathrm{Re}\left(e^{(-2+4i)t}\begin{pmatrix}1\\-4i\end{pmatrix}\right) = e^{-2t}\begin{pmatrix}\cos 4t\\ 4\sin 4t\end{pmatrix}$$

$$\mathrm{Im}\left(e^{(-2+4i)t}\begin{pmatrix}1\\-4i\end{pmatrix}\right) = e^{-2t}\begin{pmatrix}\sin 4t\\ -4\cos 4t\end{pmatrix}$$

$\Downarrow$

Gen. solution is

$$x(t) = \boxed{C_1 e^{-2t}\begin{pmatrix}\cos 4t\\ 4\sin 4t\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}\sin 4t\\ -4\cos 4t\end{pmatrix}} = \begin{pmatrix}C_1 e^{-2t}\cos 4t + C_2 e^{-2t}\sin 4t\\ 4C_1 e^{-2t}\sin 4t - 4C_2 e^{-2t}\cos 4t\end{pmatrix}$$

(b) since $\lim\limits_{t\to+\infty} e^{-2t}\cos 4t = \lim\limits_{t\to+\infty} e^{-2t}\sin 4t = 0$ then

$\boxed{x(t) \xrightarrow[t\to+\infty]{} 0 \text{ independently of the initial conditions}}$

(c) $\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix} \iff$ $C_1 = -3$

$-4C_2 = 2 \Rightarrow C_2 = -\frac{1}{2} \Rightarrow$

$$x(t) = -3e^{-2t}\begin{pmatrix}\cos 4t\\ 4\sin 4t\end{pmatrix} - \frac{1}{2}e^{-2t}\begin{pmatrix}\sin 4t\\ -4\cos 4t\end{pmatrix} =$$

$$= \boxed{\begin{pmatrix}e^{-2t}(-3\cos 4t - \frac{1}{2}\sin 4t)\\ e^{-2t}(-12\sin 4t + 2\cos 4t)\end{pmatrix}}$$

Problem 2

$$x_1' = -3x_2 + 6x_3$$
$$x_2' = 4x_1 + 5x_2 + 4x_3$$
$$x_3' = x_1 - 7x_2 - 5x_3$$

a) 1) Find the eigenvalues of $A = \begin{pmatrix} 0 & -3 & 6 \\ 4 & 5 & 4 \\ 1 & -7 & -5 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -3 & 6 \\ 4 & 5-\lambda & 4 \\ 1 & -7 & -5-\lambda \end{vmatrix} = -\lambda\left(\underbrace{(5-\lambda)(-5-\lambda)}_{\lambda^2-25}+28\right) +$$

$$+3\left(\underbrace{-4(\lambda+5)-4}_{-4\lambda-24}\right) + 6\left(\underbrace{-28-5+\lambda}_{\lambda-33}\right) = -\lambda(\lambda^2+3) -$$

$$-12\lambda - 72 + 6\lambda - 198 = -\lambda^3 - 3\lambda - 12\lambda - 72 + 6\lambda - 198 =$$

$$= -\lambda^3 - 9\lambda - 270 = 0 \iff$$

$$\lambda^3 + 9\lambda + 270 = 0$$

$\lambda = -6$ is indeed a root (check $-216 - 54 + 270 = 0$) $\Rightarrow$

$\lambda^3 + 9\lambda + 270$ is divisible by $\lambda = -6$

Make the long division of the polynomials $\lambda^3 + 9\lambda + 270$ and $\lambda + 6$:

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 45 \\ \lambda+6 & \lambda^3 + 9\lambda + 270 \\ & -\lambda^3 + 6\lambda^2 \\ \hline & -6\lambda^2 + 9\lambda + 270 \\ & -6\lambda^2 - 36\lambda \\ \hline & 45\lambda + 270 \\ & 45\lambda + 270 \\ \hline & 0 \end{array}$$

So $\lambda^3 + 9\lambda + 270 = (\lambda+6)(\lambda^2 - 6\lambda + 45)$

$$\lambda^2-6\lambda+45=\underbrace{(\lambda-3)^2+36}_{\text{completing squares}}=0 \Rightarrow \lambda_{1,2}=3\pm 6i$$

So the eigenvalues are $\lambda_{1,2}=3\pm 6i$ and $\lambda_3=-6$

2) Find an eigenvector corresponding to $\lambda_1=3+6i$

$$(A-(3+6i)I)v=0 \Leftrightarrow \begin{pmatrix} -3-6i & -3 & 6 \\ 4 & 2-6i & 4 \\ 1 & -7 & -8-6i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3-6i & -3 & 6 & 0 \\ 4 & 2-6i & 4 & 0 \\ 1 & -7 & -8-6i & 0 \end{array}\right) \xrightarrow[R_2\to\frac{1}{2}R_1]{R_1\to\frac{1}{3}R_1} \left(\begin{array}{ccc|c} -1-2i & -1 & 2 & 0 \\ 2 & 1-3i & 2 & 0 \\ 1 & -7 & -8-6i & 0 \end{array}\right) \xrightarrow[R_3\to(1+2i)R_3+R_1]{R_2\to(1+2i)R_2+2R_1}$$

(2,2) entry: $(1+2i)(1-3i)-2=1+2i-3i+6-2=5-i$

(2,3) entry: $2(1+2i)+4=2+4i+4=6+4i$

(3,2) entry: $-7(1+2i)-1=-8-14i$

(3,3) entry: $-(8+6i)(1+2i)+2=-8-6i-16i+12+2=6-22i$

$$\rightarrow \left(\begin{array}{ccc|c} -1-2i & -1 & 2 & 0 \\ 0 & 5-i & 6+4i & 0 \\ 0 & -8-14i & 6-22i & 0 \end{array}\right) \xrightarrow{R_3\to\frac{1}{2}R_3} \left(\begin{array}{ccc|c} -1-2i & -1 & 2 & 0 \\ 0 & 5-i & 6+4i & 0 \\ 0 & -4-7i & 3-11i & 0 \end{array}\right) \xrightarrow{R_3\to(5-i)R_3+(4+7i)R_2}$$

$$(3-11i)(5-i)+(4+7i)(6+4i)=15-55i-3i-11+$$
$$+24+42i+16i-28=4-58i-4+58i=0$$

$$\left(\begin{array}{ccc|c} -1-2i & -1 & 2 & 0 \\ 0 & 5-i & 6+4i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} -(1+2i)v_1-v_2+2v_3=0 \quad (\text{Eq } 1) \\ (5-i)v_2+(6+4i)v_3=0 \Rightarrow \\ (5-i)v_2=-(6+4i)v_3 \Rightarrow \end{array}$$

Take $\boxed{v_3 = 1} \Rightarrow v_2 = -\frac{6+4i}{5-i} = -\frac{(6+4i)(5+i)}{\underbrace{(5-i)(5+i)}_{5^2+1^2}} = -\frac{(6+4i)(5+i)}{26} =$

multiply the denominator and numerator by the conjugate of numerator

$$= -\frac{(3+2i)(5+i)}{13} = -\frac{15+10i+3i-2}{13} = -\frac{13+13i}{13} = -1-i \Rightarrow \boxed{v_2 = -1-i}$$

From the (Eq 1) (see the previous page)

$$-(1+2i)v_1 + 1 + i + 2 = 0 \Rightarrow (1+2i)v_1 = 3+i \Rightarrow v_1 = \frac{3+i}{1+2i} = \frac{(3+i)(1-2i)}{1^2+2^2} =$$

$$= \frac{3+i-6i+2}{5} = \frac{5-5i}{5} = 1-i \Rightarrow \boxed{v_1 = 1-i}$$

Hence $v = \begin{pmatrix} 1-i \\ -1-i \\ 1 \end{pmatrix}$

$\Downarrow$

$$e^{(3+6i)t}\begin{pmatrix} 1-i \\ -1-i \\ 1 \end{pmatrix} = e^{3t}(\cos 6t + i\sin 6t)\begin{pmatrix} 1-i \\ -1-i \\ 1 \end{pmatrix} =$$

$$= e^{3t}\begin{pmatrix} \cos 6t + \sin 6t + i(-\cos 6t + \sin 6t) \\ -\cos 6t + \sin 6t + i(-\cos 6t - \sin 6t) \\ \cos 6t + i\sin 6t \end{pmatrix}$$

$\Downarrow$

$\mathrm{Re}\left(e^{(3+6i)t}\begin{pmatrix} 1-i \\ -1-i \\ 1 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} \cos 6t + \sin 6t \\ -\cos 6t + \sin 6t \\ \cos 6t \end{pmatrix}$ and

$\mathrm{Im}\left(e^{(3+6i)t}\begin{pmatrix} 1-i \\ -1-i \\ 1 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -\cos 6t + \sin 6t \\ -\cos 6t - \sin 6t \\ \sin 6t \end{pmatrix}$ are independent solutions

3) Find the eigenvector corresponding to the eigenvalue $\lambda = -6$

$A + 6i = \begin{pmatrix} 6 & -3 & 6 \\ 4 & 11 & 4 \\ 1 & -7 & 1 \end{pmatrix}$. Solve $(A+6I)v = 0$

$$\left(\begin{array}{ccc|c} 6 & -3 & 6 & 0 \\ 4 & 11 & 4 & 0 \\ 1 & -7 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{3}R_1} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 4 & 11 & 4 & 0 \\ 1 & -7 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to 2R_3 - R_1]{R_2 \to R_2 - 2R_1}$$

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & -13 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_2]{R_2 \to \frac{1}{13}R_2} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2v_1 - v_2 + 2v_3 = 0 \\ v_2 = 0 \end{cases} \Rightarrow \begin{cases} v_1 + v_3 = 0 \\ v_2 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = -v_3 \\ v_2 = 0 \end{cases}$$

Set $v_3 = 1 \Rightarrow v_1 = -1 \Rightarrow v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$

$$e^{-6t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ is a solution}$$

$\Downarrow$

The general solution is

$$x(t) = \boxed{c_1 e^{3t}\begin{pmatrix} \cos 6t + \sin 6t \\ -\cos 6t + \sin 6t \\ \cos 6t \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -\cos 6t + \sin 6t \\ -\cos 6t - \sin 6t \\ \sin 6t \end{pmatrix} + c_3 e^{-6t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} =$$

$$= \begin{pmatrix} e^{3t}(c_1 - c_2)\cos 6t + e^{3t}(c_1 + c_2)\sin 6t - c_3 e^{-6t} \\ -e^{3t}(c_1 + c_2)\cos 6t + e^{3t}(c_1 - c_2)\sin 6t \\ c_1 e^{3t}\cos 6t + c_2 e^{3t}\sin 6t + c_3 e^{-6t} \end{pmatrix}$$

b) $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \Leftrightarrow$

$$c_1 - c_2 - c_3 = -2$$
$$-c_1 - c_2 = 3 \Rightarrow c_2 = -c_1 - 3$$
$$c_1 + c_3 = 1 \Rightarrow c_3 = 1 - c_1$$

Substituting to the first equation:

$$c_1 + c_1 + 3 + c_1 - 1 = -2 \Rightarrow 3c_1 = -4 \Rightarrow c_1 = -\frac{4}{3}$$

$$c_2 = -c_1 - 3 = \frac{4}{3} - 3 = -\frac{5}{3}$$

$$c_3 = 1 - c_1 = 1 + \frac{4}{3} = \frac{7}{3} \Rightarrow$$

$$x(t) = -\frac{4}{3}e^{3t}\begin{pmatrix} \cos 6t + \sin 6t \\ -\cos 6t + \sin 6t \\ \cos 6t \end{pmatrix} - \frac{5}{3}e^{3t}\begin{pmatrix} -\cos 6t + \sin 6t \\ -\cos 6t - \sin 6t \\ \sin 6t \end{pmatrix} + \frac{7}{3}e^{-6t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{3t}\left(\left(-\frac{4}{3}+\frac{5}{3}\right)\cos 6t - \left(\frac{4}{3}+\frac{5}{3}\right)\sin 6t\right) - \frac{7}{3}e^{-6t} \\ e^{3t}\left(\left(\frac{4}{3}+\frac{5}{3}\right)\cos 6t + \left(-\frac{4}{3}+\frac{5}{3}\right)\sin 6t\right) \\ -\frac{4}{3}e^{3t}\cos 6t - \frac{5}{3}e^{3t}\sin 6t + \frac{7}{3}e^{-6t} \end{pmatrix}$$

c) $x(t) \underset{t\to+\infty}{\longrightarrow} 0$ if and only if $c_1 = c_2 = 0 \Rightarrow$

$$x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -c_3 \\ 0 \\ c_3 \end{pmatrix}} \quad \text{for } c_3 \in \mathbb{R}$$

d) $x(t) \underset{t\to-\infty}{\longrightarrow} 0$ if and only if $c_3 = 0 \Rightarrow$

$$x(0) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} c_1 - c_2 \\ -c_1 - c_2 \\ c_1 \end{pmatrix}}$$

for any $c_1, c_2 \in \mathbb{R}$