Homework Assignment 15 MATH 308 - solutions (in problems 2&3 several methods are presented)

Problem 1
$$\begin{cases} x_1' = 5x_1 - 2x_2 + 3x_3 \\ x_2' = -2x_1 + 2x_2 + 6x_3 \\ x_3' = x_1 + 2x_2 + 3x_3 \end{cases}$$

$$A = \begin{pmatrix} 5 & -2 & 3 \\ -2 & 2 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

1) Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ -2 & 2-\lambda & 6 \\ 1 & 2 & 3-\lambda \end{vmatrix} = (5-\lambda)\big(\underbrace{\underbrace{(\lambda-2)(\lambda-3)-12}_{\lambda^2-5\lambda-6}}_{(\lambda-6)(\lambda+1)}\big) +$

$2\big(\underbrace{2(\lambda-3)-6}_{2\lambda-12}\big) + 3\big(\underbrace{-4-2+\lambda}_{\lambda-6}\big) = (\lambda-6)\big((\lambda+1)(5-\lambda) + \underbrace{4(\lambda-6)+3(\lambda-6)}_{7(\lambda-6)}\big) =$

$= (\lambda-6)(5\lambda+5-\lambda^2-\lambda+7) = -(\lambda-6)(\underbrace{\lambda^2-4\lambda-12}_{(\lambda-6)(\lambda+2)}) = -(\lambda-6)^2(\lambda+2) = 0 \Rightarrow$

The eigenvalues are $\lambda=6$ and $\lambda=-2$

Remark: Here we (by occasion) identified one of the roots of characteristic polynomial during the calculation of the determinant. In general, it will not happen, so you will need to calculate the determinant (In our case you will get $-\lambda^3+10\lambda^2-12\lambda-72$) and then try to find one (or more) root by substituting the divisors of 72, then divide you polynomial by the linear polynomial corresponding to this root, obtain a quadratic polynomial and find its root.

2) Analyzing the eigenvalue $\lambda = 6$. The algebraic multiplicity is equal to 2

Find the eigenspace corresponding to $\lambda=6$

$$(A-6I)v = \begin{pmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

There is only one equation:

(*) $\quad -v_1 - 2v_2 + 3v_3 = 0$. Geometrically it defines

a plane (the eigenplane) $\Rightarrow$ the geometric multiplicity $= 2 =$
$=$ the algebraic multiplicity

Let us find two independent vectors in this plane.
It can be done by choosing some values of, for example, v_2 and

v_3 and finding v_1 by plugging these chosen values into (*)

For example, 1) take $\quad v_2 = 1, v_3 = 0 \overset{(*)}{\Rightarrow} -v_1 - 2 = 0 \Rightarrow v_1 = -2 \Rightarrow$

$v^1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector

2) take $\quad v_2 = 0, v_3 = 1 \Rightarrow -v_1 + 3 = 0 \Rightarrow v_1 = 3 \Rightarrow v^2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

is an eigenvector $\Rightarrow \quad e^{6t}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $e^{6t}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ are 2 independent solutions

3) $\lambda = -2$. The algebraic multiplicity is equal to 1 ($\Rightarrow$ the geometric multiplicity $= 1$ necessarily)

Find an eigenvector (since geom. multiplicity $= 1$ it is defined up to a multiplication by a constant)

$$(A - (-2)I)v = (A + 2I)v = 0 \Leftrightarrow \begin{pmatrix} 7 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 7 & -2 & 3 & 0 \\ -2 & 4 & 6 & 0 \\ 1 & 2 & 5 & 0 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ -2 & 4 & 6 & 0 \\ 7 & -2 & 3 & 0 \end{array}\right) \underset{R_3 \to R_3 - 7R_1}{\overset{R_2 \to R_2 + 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 8 & 16 & 0 \\ 0 & -16 & -32 & 0 \end{array}\right) \sim$$

$$\underset{R_2 \to \frac{1}{8}R_2}{\overset{R_3 \to R_3 + 2R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left.\begin{array}{ll} (Eq1) & v_1 + 2v_2 + 5v_3 = 0 \\ (Eq2) & v_2 + 2v_3 = 0 \end{array}\right\} \text{a line (as expected)}$$

$$\Rightarrow v_2 = -2v_3$$

Fix $v_3 = 1 \Rightarrow v_2 = -2 \overset{Eq(1)}{\Rightarrow} v_1 - 4 + 5 = 0 \Rightarrow v_1 + 1 = 0 \Rightarrow v_1 = -1 \Rightarrow$

$V^3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2 \Rightarrow e^{-2t}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is another solution independent of 2 solutions found in item 2)

4) Combining items 2) & 3) we have

$$\begin{pmatrix} -2e^{6t} & 3e^{6t} & -e^{-2t} \\ e^{6t} & 0 & -2e^{-2t} \\ 0 & e^{6t} & e^{-2t} \end{pmatrix}$$ is a fundamental matrix

and the general solution is

$$x(t) = C_1 e^{6t}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{6t}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Problem 2

$$\begin{cases} x_1' = -13x_1 + 25x_2 \\ x_2' = -9x_1 + 17x_2 \end{cases}$$

a) Find the general solution

$$A = \begin{pmatrix} -13 & 25 \\ -9 & 17 \end{pmatrix}$$

1) Eigenvalues

$$\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A \lambda + \det A = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$$

$$\left(\operatorname{tr} A = -13+17 = 4, \det A = -13\cdot 17 + 25\cdot 9 = -221+225 = 4\right)$$

$\Rightarrow$ $\lambda = 2$ is the unique eigenvalue and it has the algebraic multiplicity 2.

2) Find an eigenspace of $\lambda = 2$

$$(A-2I)v = 0 \Rightarrow \begin{pmatrix} -15 & 25 \\ -9 & 15 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

Only one indep. equation

$$-3v_1 + 5v_2 = 0 \Rightarrow 3v_1 = 5v_2 \Rightarrow$$

The eigenspace is 1-dim $\Rightarrow$ geometric multiplicity $= 1 <$ alg. multiplicity and the eigenline $E_2 = C\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Now I would like to demonstrate all 3 methods discussed in class to solve this problem.

Way one

According to the general theory we know that the space $E_2^{(2)}$ of all generalized eigenvectors of order 2 is the whole plane $\mathbb{R}^2$.

1. i) Take any $w \in \mathbb{R}^2$ which is not an eigenvector, i.e. not of the form $c\begin{pmatrix}5\\3\end{pmatrix}$. For example, $w=\begin{pmatrix}1\\0\end{pmatrix}$

1. ii) Let $v=(A-2I)w=\begin{pmatrix}-15 & 25\\-9 & 15\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-15\\-9\end{pmatrix}$

(note that from the general theory v is an eigenvector

indeed $v=-3\begin{pmatrix}5\\3\end{pmatrix}$ so $v\in E_2$, found before)

1. iii) Find the representation of the matrix A in the basis (v,w) : $Av=2v$ (because v is an eigenvector corresponding to 2)

$Aw=v+2w$ (as a consequence of the fact that $(A-2I)w=v$).

Therefore the matrix B representing A in the basis (v,w) is $B=\begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}$ and $e^{tB}=e^{2t}\begin{pmatrix}1 & t\\0 & 1\end{pmatrix} \Rightarrow$

A fundamental matrix is $(v\,w)e^{tB}=$

$$=(v\,w)e^{2t}\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}=e^{2t}\left(v,\ tv+w\right)=$$

$$=e^{2t}\begin{pmatrix}-15 & -15t+1\\-9 & -9t\end{pmatrix}\Rightarrow$$

The general solution is $e^{2t}(c_1 v + c_2(tv+w))$, i.e.

$$\boxed{x(t) = e^{2t}\left(c_1\begin{pmatrix}-15\\-9\end{pmatrix} + c_2\begin{pmatrix}-15t+1\\-9t\end{pmatrix}\right)}$$

<u>Way 2</u>

2.i) Fix some eigenvector, for example, $v = \begin{pmatrix}5\\3\end{pmatrix}$ and

find w such that $(A-2I)w = v$, i.e

$$\begin{pmatrix}-15 & 25\\-9 & 15\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} \Leftrightarrow \text{one relation}$$

$$-15w_1 + 25w_2 = 5 \Leftrightarrow -3w_1 + 5w_2 = 1 \quad (**)$$

There are many solutions of this equation but we need just to choose one of them.

For this fix, for example, some w_2 and find w_1 by plugging it into $(**)$: For example take $w_2 = 0 \overset{(**)}{\Rightarrow} -3w_1 = 1 \Rightarrow w_1 = -\frac{1}{3} \Rightarrow$

$$w = \begin{pmatrix}-\frac{1}{3}\\0\end{pmatrix}$$

2.ii) Repeat the steps 1.iii) of the previous page with v and w obtained here $\Rightarrow$

The general solution

$$\boxed{x(t) = e^{2t}\left(\tilde{c}_1\begin{pmatrix}5\\3\end{pmatrix} + \tilde{c}_2\begin{pmatrix}5t-\frac{1}{3}\\3t\end{pmatrix}\right)}$$

<u>Rem</u>: Note that we obtain the same set of solutions as in the

previous method: the relation between constants (C_1, C_2) and $(\widetilde{C}_1, \widetilde{C}_2)$ is $\widetilde{C}_1 = -3C_1$

$\widetilde{C}_2 = -3C_2$

Way 3 (Using the fact that $(A-2I)^2 = 0$ ($\Leftrightarrow$ $E_2^{(2)} = \mathbb{R}^2$)

We can calculate e^{At} (without trying to find a basis of eigenvectors):

$A = 2I + (A-2I)$ and $2I$ commutes with $(A-2I)$

$$\Rightarrow \quad e^{At} = e^{2t} e^{(A-2I)t} = e^{2t} \Big(I + (A-2I)t + \underbrace{\frac{(A-2I)^2}{2!}t^2 + \dots}_{0}\Big) =$$

$$= e^{2t}\big(I + (A-2I)t\big) = e^{2t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -15 & 25 \\ -9 & 15 \end{pmatrix} t\right) =$$

$$= e^{2t}\begin{pmatrix} 1-15t & 25t \\ -9t & 1+15t \end{pmatrix} \Rightarrow \text{gen solution is}$$

$$\boxed{x(t) = e^{2t}\left(\widetilde{\widetilde{C}}_1 \begin{pmatrix} 1-15t \\ -9t \end{pmatrix} + \widetilde{\widetilde{C}}_2 \begin{pmatrix} 25t \\ 1+15t \end{pmatrix}\right)}$$

Rem Note that it defines the same set of solutions as in the 2 previous ways. For example the relation between constants $(\widetilde{C}_1, \widetilde{C}_2)$ of the answer of method 2 and constants $(\widetilde{\widetilde{C}}_1, \widetilde{\widetilde{C}}_2)$ is

$$\widetilde{C}_1 = \frac{1}{3}\widetilde{\widetilde{C}}_2$$

$$\widetilde{C}_2 = -3\widetilde{\widetilde{C}}_1 + 5\widetilde{\widetilde{C}}_2 \quad \text{(check it!)}$$

(b) If $t \to -\infty$ then $x(t) \to 0$ because $\lim_{t\to-\infty} e^{2t} = 0$

and $\lim_{t\to-\infty} te^{2t} = 0$

(c) $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

I will use the answer obtain by way 2 of a):

$$5\tilde{C}_1 - \frac{\tilde{C}_2}{3} = -4$$

$$3\tilde{C}_1 = 3 \Rightarrow \tilde{C}_1 = 1 \Rightarrow 5 - \frac{C_2}{3} = -4 \Rightarrow \frac{C_2}{3} = 9 \Rightarrow C_2 = 27 \Rightarrow$$

$$x(t) = e^{2t}\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix} + 27\begin{pmatrix} 5t - \frac{1}{3} \\ 3t \end{pmatrix}\right) = \boxed{e^{2t}\begin{pmatrix} 135t - 4 \\ 3(1+27t) \end{pmatrix}}$$

Problem 3

$$\begin{cases} x_1' = x_2 + 2x_3 \\ x_2' = 3x_1 + 2x_2 + 2x_3 \\ x_3' = -2x_1 - 2x_2 - 3x_3 \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 2 \\ -2 & -2 & -3 \end{pmatrix}$$

1) Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 2 \\ 3 & 2-\lambda & 2 \\ -2 & -2 & -3-\lambda \end{vmatrix} = -\lambda\big(\underbrace{(2-\lambda)(-3-\lambda)+4}_{\lambda^2-2\lambda+3\lambda-6+4}\big) -$

$$-\big(\overbrace{3(-3-\lambda)+4}^{-3\lambda-5}\big) + 2\big(-6+2(2-\lambda)\big) = -\lambda^3 - \lambda^2 + 2\lambda + 3\lambda + 5 -$$

$$-4\lambda - 4 = -\lambda^3 - \lambda^2 + \lambda + 1 = -\lambda^2(\lambda+1) + (\lambda+1) = -(\lambda^2-1)(\lambda+1) =$$

$$= -(\lambda-1)(\lambda+1)(\lambda+1) = -(\lambda-1)(\lambda+1)^2$$

$\Rightarrow$ Eigenvalues are $\lambda = -1$ and $\lambda = 1$

2) Analyzing $\lambda = -1$: algebraic multiplicity $= 2$

What is geometric multiplicity?

Find an eigenspace: by solving

$$(A - (-1)I)v = 0 \iff \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 3 & 2 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right) \underset{R_3 \to R_3 + 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \underset{R_3 \to R_3 + \frac{1}{2}R_2}{\overset{R_2 \to -\frac{1}{4}R_2}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{cases} v_1 + v_2 + 2v_3 = 0 \\ v_3 = 0 \end{cases} \to 2 \text{ independent equations} \Rightarrow$$

intersection of

$\iff \begin{cases} v_1 + v_2 = 0 \\ v_3 = 0 \end{cases}$ (two different planes through the origin) $\Rightarrow$ a line $\Rightarrow$

$\Rightarrow$ geom. multiplicity $= 1 <$ algebraic multiplicity

The eigenline E_{-1} is the set of all vectors of type

$$\begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Below I demonstrate two methods of solution discussed in class.

<u>Way one</u> (within analysis of $\lambda = -1$)

1i) Find the generalized eigenspace of order 2

By solving $\underbrace{(A-(-1)I)}_{(A+I)}{}^2 w = 0 \Rightarrow$

$$(A+I)^2 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1+3-4 & 1+3-4 & 2+2-4 \\ 3+9-4 & 3+9-4 & 6+6-4 \\ -2-6+4 & -2-6+4 & -4-4+4 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & 8 \\ -4 & -4 & -4 \end{pmatrix} \Rightarrow$$

$(A+I)^2 w = 0 \Leftrightarrow$ only one equation

$$w_1 + w_2 + w_3 = 0 \quad (***)$$

1ii) Choose any w satisfying $(***)$ which is not an eigenvector. (In other words any $w \in E_{-1}^{(2)}$ but not E_{-1}). For example take $w = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(it satisfies $(***)$ but not parallel to $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$)

1 iii) Calculate $v^1 = (A+I)\underset{\sim}{w}$ chosen in the previous step

$$v = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 3-2 \\ -2+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(note that according to the general theory v must be an eigenvector and this is indeed the case)

Keep vectors v^1 and w

<u>Way 2</u> (within analyzis of $\lambda=-1$)

Fix some eigenvector corresponding to $\lambda=-1$, for example $v' = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (see the end of page 9 for the eigenline of $\lambda=-1$) and find w s.t

$$(A+I)w = v', \text{ i.e.}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 3 & 3 & 2 & 1 \\ -2 & -2 & -2 & 0 \end{array}\right) \underset{R_3 \to R_3 + 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 2 & -2 \end{array}\right) \underset{R_3 \to R_3 + \frac{1}{2}R_2}{\overset{R_2 \to -\frac{1}{4}R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$\begin{cases} w_1+w_2+2w_3=-1 \\ w_3=-1 \end{cases} \Leftrightarrow \begin{cases} w_1+w_2-2=-1 \\ w_3=-1 \end{cases} \Leftrightarrow \begin{cases} w_1+w_2=1 \\ w_3=-1 \end{cases}$$

Again we have a freedom here. If we take $w_1=0 \Rightarrow$ $w_2=1$ and we get $w=\begin{pmatrix}0\\1\\-1\end{pmatrix}$

So we got the same v^1 & w as in <u>way one</u>

(again we have freedom in both method, the fact we got the same v^1 and w is just because in both cases we made the choices that matched each other)

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From the analysis of $\lambda=-1$ (in both methods) we got the following two independent solutions

$e^{-t}v^1$ & $e^{-t}(tv^1+w)$ i.e. $e^{-t}\begin{pmatrix}-1\\1\\0\end{pmatrix}$ &

$$e^{-t}\left(t\begin{pmatrix}-1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$$

3) It remains to analyze the eigenvalue $\lambda=1$.
Algebraic multiplicity is 1 ($\Rightarrow$ geometric multiplicity $=1$)

$$(A - I)v = 0 \iff \begin{pmatrix} -1 & 1 & 2 \\ 3 & 1 & 2 \\ -2 & -2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ -2 & -2 & -4 & 0 \end{array}\right) \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 + 3R_1}{\sim}} \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_2}{\overset{R_2 \to \frac{1}{4}R_1}{\sim}}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} -v_1 + v_2 + 2v_3 = 0 \\ v_2 + 2v_3 = 0 \end{array}$$

Take $v_3 = 1 \Rightarrow \begin{cases} -v_1 + v_2 + 2 = 0 \\ v_2 + 2 = 0 \Rightarrow v_2 = -2 \Rightarrow -v_1 - 2 + 2 = 0 \Rightarrow \end{cases}$

$v_1 = 0 \Rightarrow v^2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 1 \Rightarrow$

$e^t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is another solution of our system

independent of two solutions which were already found $\Rightarrow$ general solution is

$$\boxed{x(t) = C_1 e^{-t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) + C_3 e^t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} =}$$