Homework 16 MATH308, Solution of bonus $\boxed{\text{problem 3 a}}$ (three ways)

$$X' = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{A} X$$

a) Eigenvalues:

$$\det(A-\lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} = (1-\lambda)\left((2-\lambda)(-\lambda)+1\right) =$$

$$= (1-\lambda)\underbrace{(-2\lambda+\lambda^2+1)}_{(\lambda-1)^2} = -(\lambda-1)^3 = 0 \Rightarrow$$

$\lambda = 1$ is an eigenvalue of algebraic multiplicity 3

b) Find geom multiplicity of $\lambda=1$ by analyzing the system

$$(A-I)v = 0 \iff \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\begin{cases} 2v_1+v_2-v_3=0 \\ v_2-v_3=0 \end{cases} \Updownarrow \begin{cases} v_1=0 \\ v_2=v_3 \end{cases}$$

$\rightarrow$ 2 indep equation – intersection of two different planes through the origin = a line $\Rightarrow$

$\Rightarrow$ geom multiplicity $= 1$ ($<$ alg. multiplicity

c) Now I will present all three methods of solutions for this type of problems discussed in class

Way one

c 1.i) In this case the space $E_\lambda^{(2)}$ of generalized eigenvectors of order 2 has dimension 2 and the space $E_\lambda^{(3)}$ of generalized eigenvector of order 3 has dimension 3, i.e it is the whole $\mathbb{R}^3$ $(\Leftrightarrow (A-I)^3 = 0$ (check it)

Choose any w which is not in $E_\lambda^{(2)}$, i.e such that $(A-I)^2 w \neq 0$ For this we need to calculate

$$(A-I)^2 w : \quad (A-I)^2 = (A-I)(A-I) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2\cdot 0 + 1\cdot 2 + (-1)\cdot 0 & 2\cdot 0 + 1\cdot 1 + (-1)\cdot 1 & 2\cdot 0 + 1\cdot(-1) + (-1)\cdot(-1) \\ 0\cdot 0 + 1\cdot 2 + (-1)\cdot 0 & 0\cdot 0 + 1\cdot 1 + (-1)\cdot 1 & 0\cdot 0 + 1\cdot(-1) + (-1)\cdot(-1) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \Rightarrow (A-I)^2 w = 0 \Leftrightarrow 2w_1 = 0 \Leftrightarrow$$

$w_1 = 0$ → this is the equation for the plane $E_\lambda^{(2)}$

Take any $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ s.t. $w_1 \neq 0$, for exemple

$$w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

c 1.ii) Form a chain of vectors by finding w^1 and v such that: $w^1 = (A-2I)\underbrace{w}_{\text{chosen in the previous step}}$

$$v = (A-2I)w^1 = (A-2I)^2 w$$

$$w^1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Rem Note that from the general theory w^1 belongs to $E_\lambda^{(2)}$ (indeed its first component is zero) but not an eigenvector (not in E_λ) and v is an eigenvector (you can check it satisfies the system for eigenvectors $\begin{cases} v_1 = 0 \\ v_2 = v_3 \end{cases}$ as in page 6)

c 1 iii) Take a basis (v, w^1, w). Find the matrix B which represents A in this basis

$Av = v$ (as an eigenvector of $\lambda = 1$)

$Aw' = v + w'$ (because $v = (A-I)w' = Aw' - w'$)

$Aw = w' + w$ (because $w' = (A-I)w = Aw - w$

$$\begin{aligned} Av &= v \\ Aw' &= v + w' \\ Aw &= w' + w \end{aligned} \Rightarrow B = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{Jorden block} \\ \text{of size 3 for } \lambda = 1}} \Rightarrow e^{tB} = e^t \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v & w' & w \end{pmatrix} e^t \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \left(e^t v, \quad e^t (tv + w'), \; e^t \left(\frac{t^2}{2} v + tw' + w \right) \right)$$

is the fundamendel matrix of our system $\Rightarrow$

gen solution $= e^t \left(c_1 v + c_2 (tv + w') + c_3 \left(\frac{t^2}{2} v + tw' + w \right) \right)$

$$= e^t \left(c_1 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right) + c_3 \left(t^2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \right)$$

$$= e^t \begin{pmatrix} c_3 \\ c_3 t^2 + 2(c_2 + c_3) t + 2(c_1 + c_2) \\ c_3 t^2 + 2c_2 t + 2c_1 \end{pmatrix}$$

<u>Way 2</u> Here instead of starting with w which in $E_\lambda^{(3)}$ $(=\mathbb{R}^2)$ but not in $E_\lambda^{(2)}$ we fix some eigenvector v and then solve the chain of systems

$$(A-I)w^1 = v \quad (\text{for } w^1)$$
$$(A-I)w = w^1 \quad (\text{for } w)$$

c 2 i) Fix an eigenvector v. The eigenvectors satisfy the system $\begin{cases} v_1 = 0 \\ v_2 = v_3 \end{cases}$ (see page 6)

Take, for example, $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector

c 2 ii) Find w^1 such that

$$(A-I)w^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \Rightarrow \text{If } w^1 = \begin{pmatrix} w_1^1 \\ w_2^1 \\ w_3^1 \end{pmatrix} \text{ then}$$

$$\begin{cases} 2w_1^1 + w_2^1 - w_3^1 = 1 \\ w_2^1 - w_3^1 = 1 \end{cases} \Rightarrow \begin{cases} w_1^1 = 0 \quad (\ast\ast\ast) \\ w_2^1 = w_3^1 + 1 \end{cases}$$

As expected, we have infinite many solutions here but we need only one of them

We can fix w_3' and then find w_2' from the second equation of (***). For example ~~take~~ $w_3' = 0 \Rightarrow$

$$w_2' = 1 \Rightarrow \underline{w}' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

c3 iii) Find w such that

$$(A - I) w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

If $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ then this is equivalent to

$$\begin{cases} 2w_1 + w_2 - w_3 = 1 \\ w_2 - w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_1 = \frac{1}{2} \\ w_2 = w_3 \end{cases}$$

Again here we have many choices. Take for example $w_3 = 0 \Rightarrow w_2 = 0 \Rightarrow w = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$

c3 iv) Repeat the procedure of way 1 (see subitem c1 iii) with the obtained basis (v, w', w)

(note that the vectors of the basis obtained here is exactly $\frac{1}{2}$ of the vector obtained in way one)

In this way you will get the answer

$$\boxed{e^t\left(\tilde{c}_1\begin{pmatrix}0\\1\\1\end{pmatrix}+\tilde{c}_2\left(t\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}\right)+\tilde{c}_3\left(t^2\begin{pmatrix}0\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\\0\\0\end{pmatrix}\right)\right)}$$

which defines the same set of solutions as the answer obtained in way 1 (with $\tilde{c}_1=2c_1$, $\tilde{c}_2=2c_2$, $\tilde{c}_3=2c_3$)

<u>Way 3</u> Using the fact that $(A-I)^3=0$ $(\Leftrightarrow E_\lambda^{(3)}=\mathbb{R}^3)$

We can calculate e^{At} directly:

$$e^{tA}=e^te^{t(A-I)}=e^t\Big(I+(A-I)t+(A-I)^2\frac{t^2}{2}+$$

$$+\underbrace{(A-I)^3\frac{t^3}{6}+\dots}_{0,\text{ because }(A-I)^3=0}\Big)=e^t\left(I+(A-I)t+(A-I)^2\frac{t^2}{2}\right)=$$

$$=e^t\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\2&1&-1\\0&1&-1\end{pmatrix}t+\begin{pmatrix}0&0&0\\2&0&0\\2&0&0\end{pmatrix}\frac{t^2}{2}\right)=$$

$$=e^t\begin{pmatrix}1&0&0\\2t+t^2&1+t&-t\\t^2&t&1-t\end{pmatrix}.$$ This is a fundamental matrix

Therefore the general solution is

$$e^{t}\left(\tilde{\tilde{C}}_1\begin{pmatrix}1\\2t+t^2\\t^2\end{pmatrix}+\tilde{\tilde{C}}_2\begin{pmatrix}0\\1+t\\t\end{pmatrix}+\tilde{\tilde{C}}_3\begin{pmatrix}0\\-t\\1-t\end{pmatrix}\right)=$$

$$=\boxed{e^{t}\begin{pmatrix}\tilde{\tilde{C}}_1\\ \tilde{\tilde{C}}_1t^2+(2\tilde{\tilde{C}}_1+\tilde{\tilde{C}}_2-\tilde{\tilde{C}}_3)t+\tilde{\tilde{C}}_2\\ \tilde{\tilde{C}}_1t^2+(\tilde{\tilde{C}}_2-\tilde{\tilde{C}}_3)t+\tilde{\tilde{C}}_3\end{pmatrix}}$$

<u>Remark</u> It defines the same set of solutions as the answer of way one (see page 9) with

$$\begin{cases}C_1=\frac{1}{2}\tilde{\tilde{C}}_3\\ C_2=\frac{1}{2}(\tilde{\tilde{C}}_2-\tilde{\tilde{C}}_3)\\ C_3=\tilde{\tilde{C}}_1\end{cases}\qquad\text{(check!)}$$