Homework 16, MATH 308

Solution of bonus problem 3(b) (with discussion)

Problem 3(b)

$$\begin{cases} x_1' = -9x_1 + x_2 - 2x_3 \\ x_2' = x_1 - 9x_2 + 2x_3 \\ x_3' = x_1 - x_2 - 6x_3 \end{cases} \Rightarrow A = \begin{pmatrix} -9 & 1 & -2 \\ 1 & -9 & 2 \\ 1 & -1 & -6 \end{pmatrix}$$

The eigenvalues are already given: $\lambda = -8$ is the only eigenvalue and it is of algebraic multiplicity 3 ($\Rightarrow \det(A - \lambda I) = -(\lambda+8)^3$)

a) Find the geometric multiplicity of $\lambda = -8$ by analyzing the system $(A + 8I)v = 0 \Leftrightarrow$

$$-\begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix} v = 0 \Leftrightarrow$$ only one equation $v_1 - v_2 + 2v_3 = 0$ (*)

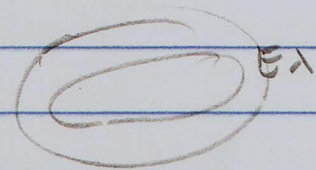
(all rows of this matrix $A + 8I$ are multiples of one row) $\Rightarrow$ the eigenvectors constitute a plane (given by equation (*)) $\Rightarrow$ geometric multiplicity $= 2$ ($<$ algebraic multiplicity)

(but still, it is different from 3a

c) Now I present several ways discussed in class

<u>Way 1</u> We know that $\dim E_\lambda = 2$ and the space $E_\lambda^{(2)}$ of generalized eigenvectors of order 2 is larger than E_λ $\left(E_\lambda \subsetneq E_\lambda^{(2)}\right) \Rightarrow \dim E_\lambda^{(2)} = 3 \Leftrightarrow$

$E_\lambda^{(2)} = R^3$.



c 1i) Choose any w which is not in E_λ

i.e does not satisfies the equation (*) for eigenvectors, i.e. (in terms of components of w)

such that $\quad w_1 - w_2 + 2w_3 \neq 0$

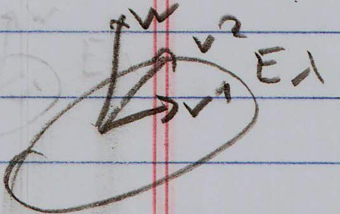
For example, take $w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

c 1ii) Calculate $v^1 = (A + 8I)w$:

$$v^1 = \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(note that according to general theory v^1 is an eigenvector, indeed its component satisfy the equation $\quad v_1 - v_2 + 2v_3 = 0$ (check: $-1-1+2=0$))

c1iii) However we still have chosen only 2 vectors v^1 & w. We need to complete them to a basis in $\mathbb{R}^3$. Remember that $\dim E_\lambda = 2$ so we still can choose an eigenvector v^2 which is not a multiple of v^1



For this return again to the equation for eigenvectors: $v_1 - v_2 + 2v_3 = 0$

Take for example $v_3 = 0 \Rightarrow v_1 = v_2$. Take $v_2 = 1 \Rightarrow$ $v_3 = 1 \Rightarrow v^2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector and it is not a multiple of $v^1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

c1iv) Take a basis (v^1, w, v^2). Find the matrix B which represents A in this basis:

$Av^1 = -8v^1$ (because v^1 is an eigenvector of $\lambda = -8$)

$Aw = v^1 - 8w$ (because $v^1 = (A+8I)w = Aw + 8w$

$Av^2 = -8v^2$ (because v^2 is an eigenvector of $\lambda = -8$)

$$\Rightarrow \quad B = \left(\begin{array}{cc|c} -8 & 1 & 0 \\ 0 & -8 & 0 \\ \hline 0 & 0 & -8 \end{array}\right) \rightarrow$$ block-diagonal with blocks being Jordan blocks

$$B = \left(\begin{array}{cc|c} -8 & 1 & 0 \\ 0 & -8 & 0 \\ \hline 0 & 0 & \boxed{-8} \end{array}\right) \Rightarrow e^{tB} = \left(\begin{array}{c|c} e^{tB_1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0\ 0 & e^{-8t} \end{array}\right) =$$

$$= \left(\begin{array}{cc|c} e^{-8t} & te^{-8t} & 0 \\ 0 & e^{-8t} & 0 \\ \hline 0 & 0 & e^{-8t} \end{array}\right) = e^{-8t} \left(\begin{array}{cc|c} 1 & t & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(v^1\ w\ v^2\right) e^{-8t} \left(\begin{array}{cc|c} 1 & t & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(e^{-8t} v^1, e^{-8t}(tv^1 + w), e^{-8t} v^2\right)$$

is a fundamental matrix $\Rightarrow$

gen. solution $x(t) = e^{-8t}\left(c_1 v^1 + c_2(tv^1 + w) + c_3 v^2\right) =$

$$= e^{-8t}\left(c_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) =$$

$$= e^{-8t} \begin{pmatrix} (-c_1 + c_2 + c_3) - c_2 t \\ (c_1 + c_3) + c_2 t \\ c_1 + c_2 t \end{pmatrix}$$

Rem The analog of algorithm 2 (i.e when we fix some $v^1 \in E_\lambda$ and try to find some w such that $(A-\lambda I)w = v^1$) will not properly work, because for randomly chosen $v^1 \in E_\lambda$ the system $(A-\lambda I)w = v^1$ will not have a solution. We have to choose from a special line in E_λ (which is generated by $\begin{pmatrix}-1\\1\\1\end{pmatrix}$) because, in general, $(A-\lambda I)w = \begin{pmatrix}-1 & 1 & -2\\ 1 & -1 & 2\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} =$

$= (w_1 - w_2 + 2w_3)\begin{pmatrix}-1\\1\\1\end{pmatrix}$, i.e $(A-\lambda I)w$ is a multiple of $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ for any w. So the choice of this special line of v^1's inside the plane E_λ already includes step (1i) and (1ii) of the previous method and in my opinion it does not deserve to be distinguished as a separate method.

Way 2 Using the fact that $(A+8I)^2=0$ (check!)

$(\Leftrightarrow E_\lambda^{(2)}=\mathbb{R}^3)$

We can calculate e^{At} directly:

$$e^{At}=e^{-8t}e^{(A+8I)t}=e^{-8t}\Big(I+(A+8I)t+\underbrace{(A+8I)^2}_{=0}\frac{t^2}{2}+\underbrace{}_{0}\Big)=e^{-8t}\big(I+(A+8I)t\big)=$$

$$=e^{-8t}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}-1&1&-2\\1&-1&2\\1&-1&2\end{pmatrix}t\right)=e^{-8t}\begin{pmatrix}1-t&t&-2t\\t&1-t&2t\\t&-t&1-2t\end{pmatrix}$$

This is a fundamental matrix $\Rightarrow$ the general solution

$$\text{is}=e^{-8t}\left(\tilde{c}_1\begin{pmatrix}1-t\\t\\t\end{pmatrix}+\tilde{c}_2\begin{pmatrix}t\\1-t\\-t\end{pmatrix}+\tilde{c}_3\begin{pmatrix}-2t\\2t\\2t\end{pmatrix}\right)=$$

$$=e^{-8t}\begin{pmatrix}\tilde{c}_1+(-\tilde{c}_1+\tilde{c}_2-2\tilde{c}_3)t\\\tilde{c}_2+(\tilde{c}_1-\tilde{c}_2+2\tilde{c}_3)t\\\tilde{c}_3+(\tilde{c}_1-\tilde{c}_2+2\tilde{c}_3)t\end{pmatrix}$$

Rem: It defines the same set of solutions as the answer by way one with $\tilde{c}_1=-c_1+c_2+c_3$, $\tilde{c}_2=c_1+c_3$, $\tilde{c}_3=c_1$ (check!)