Solution of homework #16, MATH 308

Problem 1

$$\begin{cases} x_1' = 3x_1 - x_2 + 4e^{2t} \\ x_2' = -x_1 + 3x_2 + 4e^{4t} \\ x_1(0) = 1, \ x_2(0) = 1 \end{cases}$$

Solution 1) Find a fundamental matrix of the corresponding homogeneous system

$$\begin{cases} x_1' = 3x_1 - x_2 \\ x_2' = -x_1 + 3x_2 \end{cases} \Rightarrow A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

i) Eigenvalues: $\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A\,\lambda + \det A = \lambda^2 - 6\lambda + 8 = 0$

$\operatorname{tr} A = 6$, $\det A = 9 - (-1)^2 = 8$

$D = 36 - 4 \cdot 8 = 36 - 32 = 4$

$\lambda_1 = \frac{6+2}{2} = 4$

$\lambda_2 = \frac{6-2}{2} = 2$ $\Rightarrow \lambda_1 = 4, \lambda_2 = 2 \to$ distinct real eigenvalues

ii) An eigenvector for $\lambda = 4$

$$(A - 4I)v = \begin{pmatrix} 3-4 & -1 \\ -1 & 3-4 \end{pmatrix} v = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v = 0 \Leftrightarrow v_1 + v_2 = 0 \Leftrightarrow v_1 = -v_2$$

Take $v_2 = 1 \Rightarrow v_1 = -1 \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution

iii) An eigenvector for $\lambda = 2$

$$(A - 2I)v = \begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix} v = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} v = 0 \Leftrightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$$

Take $v_2 = 1 \Rightarrow v_1 = 1 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution

$$\psi(t) = \left(e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ \ e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{pmatrix}$$ is a fundamental matrix of the homogeneous system

2) We look for a solution of the nonhomogeneous system in the form $x(t) = \Psi(t)\, c(t) = \begin{pmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$

such that $c'(t)$ satisfies

$$\Psi(t)\, c'(t) = g(t), \text{ where } g(t) = \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix} \text{ is}$$

the non-homogeneous part of our original equation

$$\Rightarrow \quad c'(t) = \Psi(t)^{-1} g(t)$$

$$\Psi(t)^{-1} = \frac{1}{\det \Psi(t)} \begin{pmatrix} e^{2t} & -e^{2t} \\ e^{4t} & e^{4t} \end{pmatrix}$$

(according to the rule that if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$)

$$\det \Psi(t) = e^{4t} \cdot e^{2t} - (-e^{4t}) e^{2t} = e^{6t} + e^{6t} = 2e^{6t} \Rightarrow$$

$$\Psi(t)^{-1} = \frac{1}{2e^{6t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ e^{4t} & e^{4t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-4t} & -e^{-4t} \\ e^{-2t} & e^{-2t} \end{pmatrix} \Rightarrow$$

$$c'(t) = \Psi(t)^{-1} g(t) = \frac{1}{\cancel{2}} \begin{pmatrix} e^{-4t} & -e^{-4t} \\ e^{-2t} & e^{-2t} \end{pmatrix} \overset{2}{\cancel{4}} \begin{pmatrix} e^{2t} \\ e^{4t} \end{pmatrix} =$$

$$= 2 \begin{pmatrix} e^{-2t} - 1 \\ 1 + e^{2t} \end{pmatrix} \Rightarrow c(t) = 2 \begin{pmatrix} \int (e^{-2t} - 1)\, dt \\ \int (1 + e^{2t})\, dt \end{pmatrix} =$$

$$= \begin{pmatrix} -e^{-2t} - 2t + C_1 \\ 2t + e^{2t} + C_2 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ the general solution $x(t) = \Psi(t)C(t) =$

$$= \begin{pmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{pmatrix} \begin{pmatrix} -e^{-2t} - 2t + c_1 \\ 2t + e^{2t} + c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} + e^{4t} \end{pmatrix} + c_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3) Now determine the constant c_1 and c_2 for our initial conditions by plugging $t=0$ into the general solution and comparing it with the initial conditions:

$$\underbrace{\begin{pmatrix} -1+1 \\ 1+1 \end{pmatrix}}_{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} + c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$$

$$c_1 = 1,\ c_2 = 0 \quad \left(\text{or} \quad \begin{matrix} c_1 + c_2 = 1 \\ -c_1 + c_2 = -1 \end{matrix} \Rightarrow c_2 = 0 \Rightarrow c_1 = 1 \right)$$

$$\Rightarrow x(t) = \begin{pmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + e^{4t} + e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} + \cancel{e^{4t}} - e^{4t} \end{pmatrix} =$$

$$\boxed{\begin{pmatrix} -e^{2t} - 2te^{4t} + 2te^{2t} + 2e^{4t} \\ e^{2t} + 2te^{4t} + 2te^{2t} \end{pmatrix} = 2te^{4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2te^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Problem 2 $X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t$

Solution 1) Find a fundamental matrix of the corresponding homogeneous system

$$X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X \Rightarrow A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

i) Eigenvalues: $\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A \, \lambda + \det A = \lambda^2 - 2\lambda + 2 = (\lambda-1)^2 + 1 = 0$

$\operatorname{tr} A = 2, \det A = 2$

$\Leftrightarrow (\lambda - 1)^2 = -1 \Leftrightarrow \lambda - 1 = \pm i \Rightarrow \lambda = 1 \pm i$

ii) An eigenvector for $\lambda = 1+i$

$$(A - (1+i)I)v = \begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} v = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} v = 0$$

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{R_1 \to iR_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right) \Leftrightarrow \text{one equation } v_1 - iv_2 = 0$$

Because $i \cdot (-i) = 1$

Take $v_2 = 1 \Rightarrow v_1 = i \Rightarrow \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = (1+i) \Rightarrow$

$\operatorname{Re}\left(e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$ and $\operatorname{Im}\left(e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$ is the fundamental set of solutions

$$e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t (\cos t + i \sin t) \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \begin{pmatrix} -\sin t + i\cos t \\ \cos t + i \sin t \end{pmatrix} =$$

$$= \underbrace{e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}_{\text{Real part}} + i \underbrace{e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}_{\text{Imaginary part}}$$

$$\Rightarrow \quad \Psi(t) = \left(e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \ e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right) = e^t \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$$

is the fundamental matrix of the homogeneous system

2) We look for a solution of the nonhomogeneous equation in the form $x(t) = \Psi(t)\,c(t) = e^t \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$

such that $c'(t)$ satisfies $\Psi(t)\,c'(t) = g(t) \Leftrightarrow$

$$\cancel{e^t} \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} c'(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \cancel{e^t} \qquad (*)$$

$$c_1'(t) \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2'(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \Rightarrow$$

$$\Rightarrow \quad c_1'(t) = 0 \quad \Rightarrow \quad c_1(t) = C_1$$

$$c_2'(t) = 1 \quad \Rightarrow \quad c_2(t) = t + C_2$$

(of course you can find the inverse etc as in problem 1 But here the solution of system $(*)$ is obvious)

$$\Rightarrow \text{gen solution } x(t) = \Psi(t)\,c(t) = e^t \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} C_1 \\ t + C_2 \end{pmatrix} =$$

$$= \boxed{e^t \begin{pmatrix} t\cos t \\ t\sin t \end{pmatrix} + C_1 e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + C_2 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}$$