Homework assignment 17 Solutions MATH 308

Problem 1

(a) i) $\begin{cases} x_1' = 5x_1 + 6x_2 \\ x_2' = -5x_1 - 8x_2 \end{cases} \Rightarrow A = \begin{pmatrix} 5 & 6 \\ -5 & -8 \end{pmatrix}$

Since $\det A \neq 0$ the only critical point is the origin

To classify the critical point find the eigenvalues of A:

$\det(A - \lambda I) = \lambda^2 + 3\lambda - 10 = 0 \Rightarrow D = 9 + 40 = 49$

$\operatorname{tr} A = -3, \ \det A = 5\cdot(-8) - (-5)\cdot 6 = -40+30$

$\Rightarrow \lambda_1 = \frac{-3+7}{2} = 2$

$\lambda_2 = \frac{-3-7}{2} = -5$

→ two distinct real eigenvalues of opposite sign $\Rightarrow$ $\boxed{\text{saddle point}}$ $\Rightarrow$ $\boxed{\text{unstable}}$

ii) To sketch the phase portrait it is preferable to find eigenlines (which in this case also gives us the separatrices)

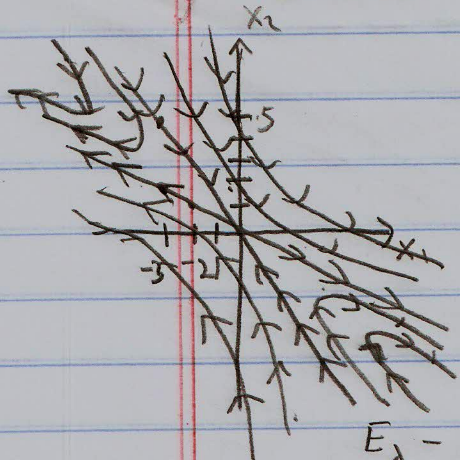
For $\lambda_1 = 2$ $\quad (A - 2I)v = \begin{pmatrix} 3 & 6 \\ -5 & -10 \end{pmatrix} v = 0 \Leftrightarrow v_1 + 2v_2 = 0 \Rightarrow$

$v_1 = -2v_2 \qquad v^1 = \begin{pmatrix} -2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \longrightarrow E_{\lambda_1} = \left\{ c \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \rightarrow$ unstable separatrix (because $\lambda_1 > 0$)

For $\lambda_2 = -5$ $\quad (A + 5I)v = \begin{pmatrix} 10 & 6 \\ -5 & -3 \end{pmatrix} v = 0 \Rightarrow 5v_1 + 3v_2 = 0 \Rightarrow$

$v_1 = -\frac{3}{5} v_2 \Rightarrow v^2 = \begin{pmatrix} -\frac{3}{5} v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -\frac{3}{5} \\ 1 \end{pmatrix} \Rightarrow E_{\lambda_2} = \left\{ c \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right\} \rightarrow$ stable separatrix (because $\lambda_2 < 0$)

i) <u>Sketch</u>:



Problem 1 (b) i) $\begin{cases} x_1' = 3x_1 + 5x_2 \\ x_2' = -4x_1 - 5x_2 \end{cases}$ $A = \begin{pmatrix} 3 & 5 \\ -4 & -5 \end{pmatrix}$, $\det A \neq 0 \Rightarrow$

the <u>origin</u> is the only critical point

To classify the critical point find the eigenvalues of A:

$\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A\,\lambda + \det A = \lambda^2 + 2\lambda + 5 =$

$\operatorname{tr} A = -2$, $\det A = -15 + 20 = 5$

$= \lambda^2 + 2\lambda + 1 + 4 = (\lambda+1)^2 + 2^2 \Leftrightarrow (\lambda+1)^2 = -2^2 \Rightarrow \lambda + 1 = \pm 2i \Rightarrow$

$\lambda = -1 \pm 2i \xrightarrow[\operatorname{Re}\lambda < 0]{\text{complex eigenvalues}}$ spiral sink $\Rightarrow$ $\boxed{\text{asymptotically stable}}$

i.2) Let us find an eigenvector of $\lambda = -1 + 2i$ in order to find the direction of rotation on the spiral trajectories

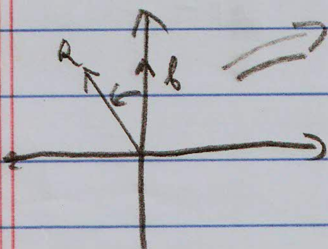
$(A - (-1+2i)I)v = \begin{pmatrix} 4-2i & 5 \\ -4 & -4-2i \end{pmatrix} v = 0 \Leftrightarrow (4-2i)v_1 + 5v_2 = 0$

(the second equation is obtained here from the first by multiplication on $-\frac{1}{5}(4+2i)$)

$$(4-2i)v_1 + 5v_2 = 0 \Rightarrow v_2 = -\frac{1}{5}(4-2i)v_1 \Rightarrow$$

$$v = \begin{pmatrix} v_1 \\ -\frac{1}{5}(4-2i)v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -\frac{1}{5}(4-2i) \end{pmatrix} =$$

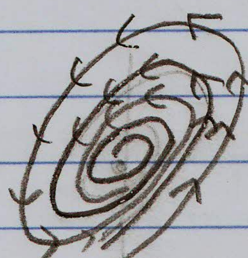
$$= v_1 \left(\underbrace{\begin{pmatrix} 1 \\ -\frac{4}{5} \end{pmatrix}}_{a} + i \underbrace{\begin{pmatrix} 0 \\ \frac{2}{5} \end{pmatrix}}_{b} \right)$$



$\Rightarrow$ rotation is <u>counterclockwise</u>

(from $\vec{b}$ to $\vec{a}$ in the shortest way

<u>Sketch</u>



Problem 1c)

$$\begin{cases} x_1' = -6x_1 + 3x_2 + 3 \\ x_2' = x_1 - 4x_2 + 10 \end{cases}$$

i) Find the critical point by solving the system of equation

$$\begin{cases} -6x_1 + 3x_2 + 3 = 0 \\ x_1 - 4x_2 + 10 = 0 \end{cases} \Leftrightarrow \begin{cases} 6x_1 - 3x_2 = 3 & \text{Eq 1} \\ x_1 - 4x_2 = -10 & \text{Eq 2} \end{cases}$$

Eq 1 − 6 Eq 2: $(-3+24)x_2 = 63 \Leftrightarrow 21x_2 = 63 \Rightarrow x_2 = 3$

$\Rightarrow x_1 = 4x_2 - 10 = 4 \cdot 3 - 10 = 2 \Rightarrow$

The critical point is $\boxed{(2,3)}$

Now consider the corresp. homogeneous system

$$\begin{cases} u_1' = -6u_1 + 3u_2 \\ u_2' = u_1 - 4u_2 \end{cases} \Rightarrow A = \begin{pmatrix} -6 & 3 \\ 1 & -4 \end{pmatrix}$$

To classify the critical point find the eigenvalues of A:

$$\det(A - \lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 + 10\lambda + 21$$

$$\operatorname{tr}A = -10,\ \det A = 21$$

$$D = 100 - 4 \cdot 21 = 100 - 84 = 16$$

$$\lambda_1 = \frac{-10+4}{2} = -3$$

$$\lambda_2 = \frac{-10-4}{2} = -7$$

two distinct negative eigenvalues $\Rightarrow$

$\boxed{(2,3) \text{ is a nodal sink, asymptotically stable}}$

ii) To have a more exact sketch it is preferable to find the eigenlines

$\lambda_1 = -7$: $(A + 7I)v = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} v = 0 \Rightarrow v_1 + 3v_2 = 0$

$\Rightarrow$ the eigenline E_{λ_1} of $\lambda_1 = -7$ is generated by $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

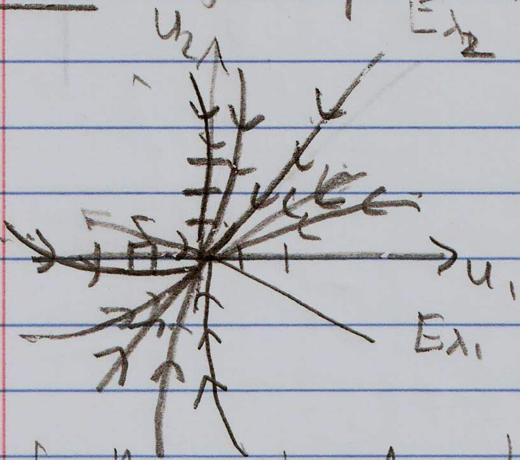
$\lambda_2 = -3$

$(A+3I)v = \begin{pmatrix}-3 & 3\\ 1 & -1\end{pmatrix} v = 0 \Rightarrow v_1 - v_2 = 0 \Rightarrow$ the eigenline

E_{λ_2} of $\lambda_2 = -3$ is generated by $\begin{pmatrix}1\\1\end{pmatrix}$

$$|\lambda_2| < |\lambda_1|$$

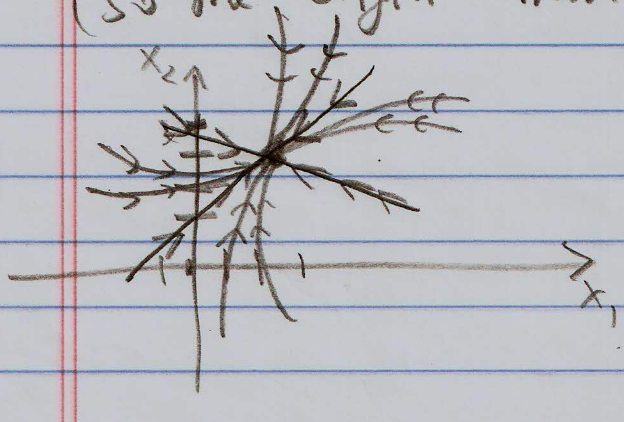
__Sketch__ of the phase portrait __for $u' = Au$__



Tangent lines of trajectories tends to the eigenline E_{λ_2} as one approaches the origin

__Sketch for the original system__

For the original system the phase phase portrait is obtained from the above by the shift in (2,3) (so the origin move to the point (2,3)



Problem 1 d) $\begin{cases} x_1' = -2x_1 + x_2 - 6 \\ x_2' = -x_1 - 4x_2 + 15 \end{cases}$

i) Find the critical point by solving the system of equations

$$\begin{cases} -2x_1 + x_2 - 6 = 0 \\ -x_1 - 4x_2 + 15 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x_1 - x_2 = -6 & Eq\,1 \\ x_1 + 4x_2 = 15 & Eq\,2 \end{cases}$$

$\Rightarrow (Eq\,1) - 2(Eq\,2) \rightarrow -9x_2 = -36 \Rightarrow x_2 = 4 \Rightarrow$
$x_1 = 15 - 4x_2 = -1$

$\Rightarrow \boxed{(-1, 4)}$ is the only critical point

Now consider the corresponding homogeneous system

$$\begin{cases} u_1' = -2u_1 + u_2 \\ u_2' = -u_1 - 4u_2 \end{cases} \Rightarrow A = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$$

To classify the critical point find the eigenvalue of A:

$$\det(A - \lambda I) = \lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0$$

$\operatorname{tr}A = -6$, $\det = 8 + 1 = 9$

$\Rightarrow$ a repeated eigenvalue $\lambda = -3 < 0$
Since $A \neq -3I$ then geom mult $= 1 <$ alg. multiplicity

$\rightarrow$ $\boxed{\text{improper nodal sink}}$ $\Rightarrow$ $\boxed{\text{asymptotically stable}}$

iii) To have a more exact sketch it is preferable to find the eigenline and to what half-plane is directed the generalised eigenvector w if we fix an eigenvector v.

Eigenline $(A+3I)v = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} v = 0 \Rightarrow v_1 + v_2 = 0$

$\Rightarrow$ The eigenspace E_λ is generated by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

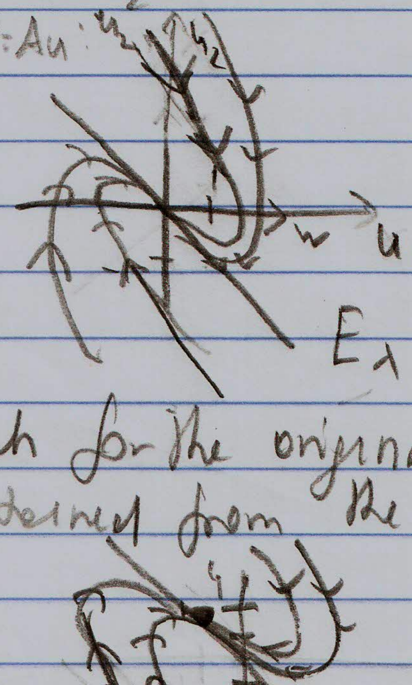
Let $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Find w s.t. $(A+3I)w = v$

$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Leftrightarrow w_1 + w_2 = 1$

Take $w_2 = 0 \Rightarrow w_1 = 1$

Sketch for $u' = Au$:



The trajectories enter the origin from the 1st and 3rd quadrat w.r.t (v, w) and their tangent tends to E_λ

Sketch for the original system: the phase portrait is obtained from the above by the shift in $(-1, 4)$