Problem 2(a) $\begin{cases} x' = 16 - xy \\ y' = x - y^3 \end{cases}$

i) To find the critical points solve the system

$$\begin{cases} 16 - xy = 0 & (\text{Eq } 1) \\ x - y^3 = 0 & (\text{Eq } 2) \end{cases}$$

(Eq 2) $\Rightarrow x = y^3$. Substituting this to (Eq 1) we get

$16 - y^4 = 0 \Rightarrow y^4 = 16 \Rightarrow y = \pm 2 \Rightarrow$

If $y = 2$ then $x = 8$
If $y = -2$ then $x = -8$ $\Rightarrow$ 2 critical points $(8,2)$ and $(-8,-2)$

ii) To find the corresponding linear systems first calculate the corresponding Jacobian matrix at any point. Let $f(x,y) = 16 - xy$, $g(x,y) = x - y^3$

Then $J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}$

ii 1) If $(x,y) = (8,2)$ then $J(8,2) = \begin{pmatrix} -2 & -8 \\ 1 & -12 \end{pmatrix}$

The corresponding linear system is

$$u' = \begin{pmatrix} -2 & -8 \\ 1 & -12 \end{pmatrix}$$

ii 2) If $(x,y) = (-8,-2)$ then

$$J(-8,-2) = \begin{pmatrix} 2 & 8 \\ 1 & -12 \end{pmatrix}$$

iii) 1) Near $(8,2)$: $J(8,2) = \begin{pmatrix} -2 & -8 \\ 1 & -12 \end{pmatrix}$

$$\det(J(8,2) - \lambda I) = \lambda^2 + 14\lambda + 32$$

$\operatorname{tr} J = -14$, $\det J = (-2)\cdot(-12) - (-8) = 24+8 = 32$

$D = 14^2 - 4\cdot 32 = \underbrace{196-128}_{=68} > 0 \rightarrow$ distinct real roots

By Vieta Thm: $\lambda_1 \lambda_2 = 16 > 0 \Rightarrow \lambda_1$ & λ_2 have the same sign

$\lambda_1 + \lambda_2 = -14 < 0 \Rightarrow$ both λ_1 and λ_2 are <u>negative</u>

$\Rightarrow$ $\boxed{\text{nodal sink}}$ $\Rightarrow$ $\boxed{(8,2) \text{ is asymptotically stable for the original system}}$

In general $\lambda_{1,2} = \dfrac{-14 \pm \sqrt{68}}{2} = -7 \pm \sqrt{17}$ (I asked for this but Vieta Thm is more useful (see Remarks on pages 13-14))

iii) 2) Near $(-8,-2)$ $J(-8,-2) = \begin{pmatrix} 2 & 8 \\ 1 & -12 \end{pmatrix}$

$$\det(J(-8,-2) - \lambda I) = \lambda^2 + 10\lambda - 32$$

$\operatorname{tr} J(-8,-2) = 10$

$\det J(-8,-2) = 2\cdot(-12) - 8 = -32$

You can immediately make classification of the critical point in this case: By Vieta theorem $\lambda_1\lambda_2 < 0 \Rightarrow$

λ_1, λ_2 are real and of opposite signs $\Rightarrow$ $\boxed{\text{saddle point}}$ $\Rightarrow$

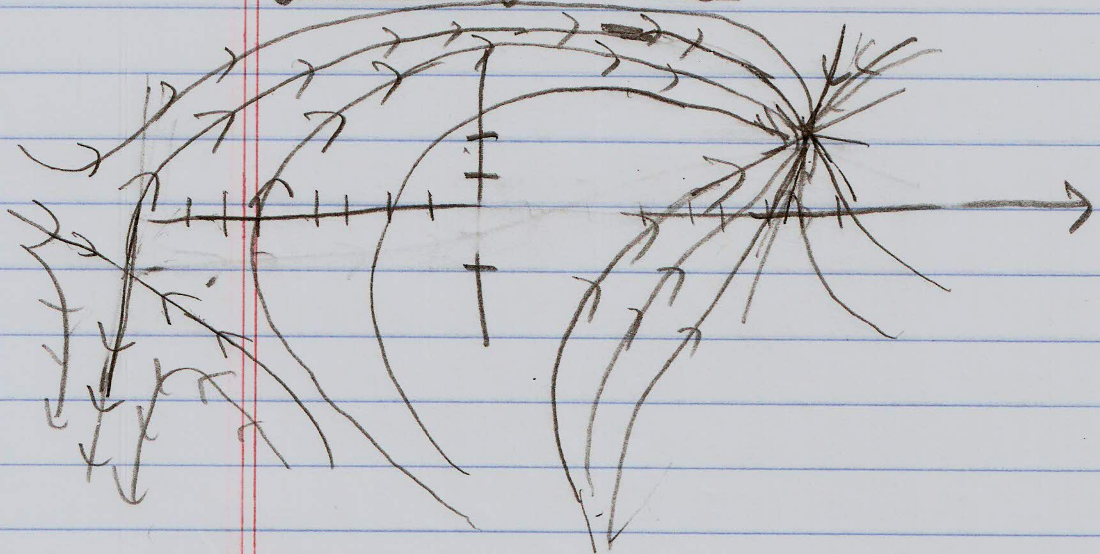
$\Rightarrow$ $(-8,-2)$ is a saddle point and $\boxed{\text{unstable}}$ for the original system

$D = 10^2 + 4\times 32 = 100 + 128 = 228$

$\lambda_{1,2} = \dfrac{-10 \pm \sqrt{228}}{2} = -5 \pm \sqrt{57}$

} I asked it but actually you can classify the point without finding λ_1 & λ_2 explicitely (as we did above)

<u>Sketch of the phase portrait for the original system</u>



Problem 2 (b) $\begin{cases} x' = (2+x)(y-x) \\ y' = (4-x)(x+y) \end{cases}$

i) To find critical points solve the system

$$\begin{cases} (2+x)(y-x) = 0 \Rightarrow \text{either } x=-2 \text{ or } y=x \\ (4-x)(x+y) = 0 \Rightarrow \text{either } x=4 \text{ or } y=-x \end{cases}$$

We have 4 possibilities

1) $x=-2$ & $x=4 \rightarrow$ impossible

2) $x=-2$ & $y=-x=2 \Rightarrow \boxed{(-2,2)}$

3) $y=x$ & $x=4 \Rightarrow y=4 \Rightarrow \boxed{(4,4)}$

4) $y=x$ & $y=-x \Rightarrow y=0 \Rightarrow x=0 \Rightarrow \boxed{(0,0)}$

ii) To find the corresponding linear systems first calculate the corresponding Jacobian matrix at any point.

$$f(x,y) = (2+x)(y-x) = 2y + xy - 2x - x^2$$
$$g(x,y) = (4-x)(x+y) = 4x - x^2 + 4y - xy$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} y-2-2x & 2+x \\ 4-2x-y & 4-x \end{pmatrix}$$

ii.1) $$J(-2,2) = \begin{pmatrix} 2-2-2(-2) & 0 \\ 4-2(-2)-2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}$$

The corresp. linear system is $u' = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix} u$

ii 2) $J(4,4) = \begin{pmatrix} 4-2-2\cdot 4 & 2+4 \\ \cancel{4}-2\cdot 4-\cancel{4} & 0 \end{pmatrix} =$

$= \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}$

The corresp. linear system is $u' = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix} u$

ii 3) $J(0,0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}$

The corresp. linear system is $u' = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} u$

iii 1) Near $(-2,2)$: $J(-2,2) = \begin{pmatrix} 4 & 0 \\ 6 & 2 \end{pmatrix} \rightarrow$ lower triangular

The eigenvalues = the elements on the diagonal $\Rightarrow$

$\lambda_1 = 4$, $\lambda_2 = 6$ $\rightarrow$ positive distinct $\rightarrow$ $\boxed{\text{nodal}}$

$\boxed{\text{source}}$ $\Rightarrow$ $\boxed{(-2,2) \text{ is unstable for the original system}}$

iii 2) Near $(4,4)$: $J(4,4) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}$

$\det(J(4,4) - \lambda I) = \lambda^2 + 6\lambda + 48$ $(tr = -6, \det = 48)$

$D = 36 - 192 < 0 \Rightarrow$ complex roots

$\lambda_{1,2} = \dfrac{-6 \pm i\sqrt{|D|}}{2} \Rightarrow \operatorname{Re}\lambda_{1,2} < 0 \Rightarrow$

[redacted]

$\Rightarrow$

$\Rightarrow$ $\boxed{\text{spiral sink}}$ $\Rightarrow$ $(4,4)$ is asymptotically stable for the original system

The eigenvalues are

$$\lambda_{1,2} = \frac{-6 \pm i\sqrt{156}}{2} = -3 \pm i\sqrt{39}$$

iii 3 Near $(0,0)$: $J(0,0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}$

$$\det\left(J(0,0) - \lambda I\right) = \lambda^2 - 2\lambda - 16$$

trace $= 2$, det $= -8-8 = -16$

$D = 4 + 64 = 68 > 0 \Rightarrow$ distinct real roots

By Vieta Thm $\lambda_1 \lambda_2 = -16 \Rightarrow \lambda_1$ and λ_2 have opposite signs $\Rightarrow$ saddle point $\Rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$ of the original system and it is $\boxed{\text{unstable}}$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{68}}{2} = 1 \pm \sqrt{17}$$

(for the final sketch see page 15)

Important remark As a matter of fact, in order to determine the type of the critical point of a linear system it is not necessary to find the eigenvalues exactly but it is enough to determine the sign of their real part

The latter is based on the analysis of the sign of the discriminant and the Vieta Thm which implies that the product of the eigenvalues $= \det A$ and the sum of eigenvalues is equal to the trace

(see next page!)