Solutions. Homework assignment 1 MATH 308-505

Problem 1 (a) If $v(t) \equiv v_e$ is a solution then

$$0 = v'(t) = 4.9 - \frac{v(t)}{40} = 4.9 - \frac{v_e}{40}, \text{ i.e.}$$

v_e has to satisfy the following equation

$$4.9 - \frac{v_e}{40} = 0 \Rightarrow v_e = 40 \cdot 4.9 = \boxed{196}$$

(b) As derived in class the equation

$$v' = av + b$$

has a general solution

$$v(t) = -\frac{b}{a} + Ce^{at}$$

In our case $b = 4.9$, $a = -\frac{1}{40} \Rightarrow -\frac{b}{a} = 196 \ (= v_e) \Rightarrow$

The general solution of our equation is

$$v(t) = 196 + Ce^{-\frac{t}{40}}$$ (you can also derive this formula directly using separation)

Determine C from the initial condition $v(0) = 0$

$$0 = 196 + C \Rightarrow C = -196 \Rightarrow$$

$$\boxed{v(t) = 196 - 196e^{-\frac{t}{40}}} = \boxed{196\left(1 - e^{-\frac{t}{40}}\right)}$$

$$\lim_{t \to +\infty} v(t) = \lim_{t \to +\infty}\left(196 - 196e^{-\frac{t}{40}}\right) = 196 - \underbrace{\lim_{t \to +\infty} 196e^{-\frac{t}{40}}}_{0} = \boxed{196} =$$

$\boxed{= v_e}$ (i.e. the limiting velocity is equal to the answer in (a)

(c) We have to solve the equation

$$196\left(1-e^{-\frac{t}{40}}\right) = 0.3 \cdot 196 \Rightarrow$$

$$1-e^{-\frac{t}{40}} = 0.3 \Rightarrow$$

$$e^{-\frac{t}{40}} = 0.7 \Rightarrow$$

$$-\frac{t}{40} = \ln\frac{7}{10} \Rightarrow t = -40\ln\frac{7}{10} = \boxed{40\ln\frac{10}{7}}$$

(d) Let $s(t)$ be the distance the object has fallen in time t $\Rightarrow s'(t) = v(t)$, $s(0)=0 \Rightarrow$

$$s\left(40\ln\frac{10}{7}\right) = \int_0^{40\ln\frac{10}{7}} v(t)\,dt = \int_0^{40\ln\frac{10}{7}} 196\left(1-e^{-t/40}\right)dt =$$

$$= 196\left(t+40e^{-t/40}\right)\Big|_0^{40\ln\frac{10}{7}} = 196\Big(40\ln\frac{10}{7} +$$

$$+ 40\underbrace{e^{-40\ln\frac{10}{7}}}_{\left(e^{\ln\frac{7}{10}}\right)^{40} = \left(\frac{7}{10}\right)^{40}} - 40\Big) = \boxed{196\left(40\ln\frac{10}{7} + 40\left(\frac{7}{10}\right)^{40} - 40\right)} =$$

$$= \boxed{7840\left(\ln\frac{10}{7} + \left(\frac{7}{10}\right)^{40} - 1\right)}$$

Problem 2

a) $y' = e^{x+y} = \underbrace{e^x e^y}_{\text{separable}}$

$e^{-y}dy = e^x dx \Rightarrow \int e^{-y}dy = \int e^x dx + C \Rightarrow$

$-e^{-y} = e^x + C_1 \Rightarrow$

$e^{-y} = -e^x - C_1 \Rightarrow$ Let $C = -C_1$

$-y = \ln(C - e^x) \Rightarrow \boxed{y = -\ln(C - e^x)}$

b) $(x^2-1)y' + 2xy^2 = 0$

$y' = \underbrace{-\frac{2x}{x^2-1}y^2}_{\text{separable}}$

$\frac{dy}{y^2} = -\frac{2x}{x^2-1}dx \Rightarrow \underbrace{\int\frac{dy}{y^2}} = -\int\frac{2x}{x^2-1}dx + C_1$

$-\frac{1}{y} = -\ln|x^2-1| + C_1$

Let $C = -C_1 \Rightarrow$ $\frac{1}{y} = \ln|x^2-1| + C \Rightarrow$

$\boxed{y = \frac{1}{\ln|x^2-1| + C}}$