Solution of homework #1 MATH 308 - FALL 2016

Problem 1

(a) $v' = 19.6 - \frac{v}{a}$

v_e satisfies the equation

$$19.6 - \frac{v_e}{a} = 0$$

(indeed, $v(t) \equiv v_e$ is a solution if and only if $0 = 19.6 - \frac{v_e}{a}$)

$$\Rightarrow \boxed{v_e = 19.6\, a}$$

(b) First find the general solution

$v' = 19.6 - \frac{v}{a}$ is separable

$$\frac{dv}{19.6 - \frac{v}{a}} = dt \Leftrightarrow \int \frac{dv}{19.6 - \frac{v}{a}} = \int dt + C_1$$

$$u = 19.6 - \frac{v}{a} \Rightarrow du = -\frac{1}{a}\, dv$$

$$\Rightarrow dv = -a\, du$$

$$\underbrace{-a \int \frac{du}{u}}_{\ln|u|} = t + C_1 \Leftrightarrow \ln\left|19.6 - \frac{v}{a}\right| = -\frac{t}{a} - \frac{C_1}{a}$$

$$\Downarrow$$

$$\left|19.6 - \frac{v}{a}\right| = \underbrace{e^{-\frac{C_1}{a}}}_{C_2} e^{-\frac{t}{a}} \Rightarrow$$

$$\Rightarrow 19.6 - \frac{v}{a} = \underbrace{\pm C_2}_{-C_3} e^{-\frac{t}{a}}$$

$$\frac{v}{a} = 19.6 + C_3 e^{-\frac{t}{a}} \Rightarrow$$

$$\boxed{v(t) = 19.6a + \underbrace{C_3 a}_{C} e^{-t/a}}$$ – the general solution

<u>Rem</u> You also could use instead the formula derived in class for $y' = \alpha y + \beta$

$$y(t) = -\frac{\beta}{\alpha} + c e^{\alpha t}$$

with $y(t) = v(t)$, $\beta = 19.6$, $\alpha = -\frac{1}{a} \Rightarrow$

$$-\frac{\beta}{\alpha} = 19.6\,a \text{ and}$$

$$v(t) = 19.6a + Ce^{-\frac{t}{a}}$$

If $v(0) = 0$ then $0 = 19.6a + C \Rightarrow C = -19.6 \cdot a$

$$\Rightarrow \boxed{v(t) = 19.6a\left(1 - e^{-\frac{t}{a}}\right)}$$

$$\lim_{t \to +\infty} v(t) = \lim_{t \to +\infty} 19.6a\left(1 - \underbrace{e^{-\frac{t}{a}}}_{\substack{\downarrow t \to +\infty \\ 0}}\right) = \boxed{19.6a} = v_e$$

(c) We have to find t such that

$$\cancel{19.6a}\left(1 - e^{-\frac{t}{a}}\right) = \frac{3}{4}\,\cancel{19.6a}$$

$$1-e^{-\frac{t}{a}} = \frac{3}{4} \Leftrightarrow e^{-\frac{t}{a}} = \frac{1}{4}$$

$$-\frac{t}{a} = \ln\frac{1}{4} = -\ln 4 \Rightarrow \boxed{t = a\ln 4}$$

(d) The distance $= \int_0^{\ln 4} v(t)\,dt = \int_0^{\ln 4} 19.6a\left(1-e^{-\frac{t}{a}}\right)dt =$

$$= 19.6a\left(t + a\cdot e^{-\frac{t}{a}}\right)\Big|_0^{a\ln 4} =$$

$$= 19.6a\left(a\ln 4 + \underbrace{a\,e^{-\frac{\cancel{a}\ln 4}{\cancel{a}}}}_{\frac{1}{4}} - a\right) =$$

$$= 19.6a^2\left(\ln 4 - \frac{3}{4}\right)$$

Problem 2

(a) $y' = \underbrace{1-t+y^2-ty^2}_{(1-t)(1+y^2)} \Rightarrow$ separable

Separate $\frac{dy}{1+y^2} = (1-t)\,dt \Rightarrow \int \frac{dy}{1+y^2} = \int (1-t)\,dt$

$$\arctan y = t - \frac{t^2}{2} + C \Rightarrow$$

$$y = \tan\left(t - \frac{t^2}{2} + C\right)$$

1(f) $(1+t^2)^{1/2}\,dy - ty^3(1+t^2)^{-1/2}\,dt = 0$

$$\frac{dy}{dt} = ty^3(1+t^2)^{-1/2}(1+t^2)^{-1/2} =$$

$$= \underbrace{\frac{t}{1+t^2}y^3}_{\text{separable}}$$

$$\Leftrightarrow \frac{dy}{y^3} = \frac{t}{1+t^2}\,dt \Leftrightarrow \int\frac{dy}{y^3} = \int\frac{t\,dt}{1+t^2} + C_1$$

$$-\frac{1}{2}y^{-2} = \frac{1}{2}\ln(1+t^2) + C_1$$

$$\Rightarrow \quad y^{-2} = -\ln(1+t^2)\underbrace{-2C_1}_{C} = -\ln(1+t^2) + C$$

$$y = \pm\frac{1}{\sqrt{C-\ln(1+t^2)}}$$

If $y(0) = 1 > 0$ then $1 = y(0) = \frac{1}{\sqrt{C-\ln 1}} = \frac{1}{\sqrt{C}} \Rightarrow C = 1$

$$\Rightarrow \boxed{y(t) = \frac{1}{\sqrt{1-\ln(1+t^2)}}}$$