Homework Assignment #1 FALL 2016 - MATH308, Honors Section

due January 26, 2017 at the beginning of class

Topics covered : equations $y' = ay + b$, where a and b are constant, the model of falling object in air resistance (section 1.2 and week 1 lecture notes), separable equations (section 2.2 and week 1 lecture notes), equations that can be reduced to separable equations by an appropriate substitution (beginning of lecture notes of week 2). *You do not need to use calculator for this assignment.*

1. The **half-life** of a radioactive material is the time required for an amount of the material to decay to one-half of its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, $r > 0$, the half-life is equal to $\frac{\ln 2}{r}$ (in particular, it does not depend on the original value $Q(0)$) .

2. Assume that the velocity v of the falling object satisfies the following differential equation:

$$v'(t) = 9.8 - \frac{v(t)}{a} \tag{1}$$

where a is a positive constant.

   (a) Find a number v_e such that $v(t) \equiv v_e$ is a solution of equation (1) (in other words find the equilibrium solution of (1)).

   (b) Solve the equation (1) with initial condition $v(0) = 4.9a$. What is the limit of this solution when $t \to +\infty$? How this limiting velocity is related to your answer in the item (a)?

   (c) Find the time (from the initial time moment is 0) that must elapse for the object to reach 90% of the limiting velocity found in the item (b).

   (d) How far does the object fall in the time found in the item (c).

   (e) For small, slowly falling object , the assumption made in the class that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity. Write a differential equation for the velocity v of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity (i.e., $|F_{drag}| = \gamma v^2$ for some positive constant γ) and its direction is opposite to that of the velocity. Assume that the direction of positive velocity is down.

3. Solve the following differential equations:

   (a) Find the general solution of $2t + ty^2 + e^{t^2} yy' = 0$ (please express y as a function of x explicitly).

   (b) Find the solution of the initial value problem $(xy^2 + x)\dfrac{dy}{dx} + x^2 y - y = 0$, $y(1) = e$.

4. Before attempting this problem review the lecture notes from week 2, discussing the equation of the type $y' = f(\frac{y}{x})$ (so-called, homogeneous equations) and $y' = f(ax + by + c)$: the main idea here is to make an appropriate substitution to obtain a separable equation: $u(x) = \frac{y(x)}{x}$ in the first case and $u(x) = ax+by(x)+c$ in the second case. Then find the general solution of the following equations:

   (a) $y' = \dfrac{y - 2x}{2x + y}$; (b) $y' = (3x - 2y - 1)^2$.

5. Consider the differential equation

$$\frac{dy}{dt} = \frac{ay + bt + m}{cy + dt + n}, \tag{2}$$

where $a, b, c, d, m,$ and n are constant.

   (a) Make a substitution of the dependent and independent variables as follows: $t = T + h, y = Y + k$ for some constants h and k. Show that if $ad - bc \neq 0$, then there exists h and k such that in the new coordinate the equation (3) is transformed to

$$\frac{dY}{dT} = \frac{aY + bT}{cY + dT}. \tag{3}$$

   (b) Based on your solution of the previous item and of the problem 3 (a) find the general solution of the equation $y' = \dfrac{y - 2x + 1}{2x + y - 3}$.

   (c) Solve the equation (3) if $ad - bc = 0$.