Homework assignment 1 Solution MATH 308 - 301/302

1) (a) v_e must satisfy the following equation

$$9.8 - \frac{v_e}{20} = 0 \Rightarrow \boxed{v_e = 196}$$

(Indeed, if $v(t) \equiv v_e$ is a solution then

$$0 = v'(t) = 9.8 - \frac{v(t)}{20} = 9.8 - \frac{v_e}{20} \Rightarrow 9.8 - \frac{v_e}{20} = 0)$$

(b) We have shown in class that a general solution of the equation

$$v' = av + b \text{ has the form}$$

$$v(t) = -\frac{b}{a} + Ce^{at}$$

If $v(0) = 0$ then $0 = -\frac{b}{a} + C \Rightarrow C = \frac{b}{a} \Rightarrow$

$$v(t) = -\frac{b}{a} + \frac{b}{a}e^{at} = \frac{b}{a}(e^{at} - 1)$$

In our situation $b = 9.8$, $a = -\frac{1}{20} \Rightarrow \frac{b}{a} = -196 \Rightarrow$

$$\boxed{v(t) = 196 - 196e^{-\frac{t}{20}} = 196(1 - e^{-t/20})}$$

$$\boxed{\lim_{t \to +\infty} v(t) = 196 = v_e}$$

(c) We have to solve the equation

$$\cancel{196}(1 - e^{-t/20}) = 0.75\ \cancel{196} \Leftrightarrow$$

$$1 - e^{-t/20} = \frac{3}{4} \Rightarrow e^{-t/20} = \frac{1}{4} \Rightarrow$$

$$-\frac{t}{20} = \ln\frac{1}{4} = -\ln 4 \Rightarrow t = 20\ln 4 = \boxed{40\ln 2}$$

d) Let $s(t)$ be the distance the object has fallen in time $t \Rightarrow s'(t) = v(t) \Rightarrow$

$$s(40\ln 2) = \int_0^{40\ln 2} v(t)\,dt = \int_0^{40\ln 2} 196\left(1 - e^{-t/20}\right)dt =$$

$$= 196\left(t + 20e^{-t/20}\right)\Big|_{t=0}^{40\ln 2} =$$

$$= 196\Big(40\ln 2 + 20\underbrace{e^{-\frac{40\ln 2}{20}}}_{e^{-2\ln 2} = 2^{-2} = \frac{1}{4}} - 20\underbrace{e^{0}}_{1}\Big) =$$

$$= 196\left(40\ln 2 + 20\cdot\frac{1}{4} - 20\right) = 196\left(40\ln 2 - 15\right)$$

2. $y' + y^2\cos x = 0$

Separate the variables

$$y' = -y^2\cos x \Leftrightarrow \frac{y'}{y^2} = -\cos x \Leftrightarrow \frac{dy}{y^2} = -\cos x\,dx$$

Integrate $\displaystyle\int \frac{dy}{y^2} = \int -\cos x\,dx + C \Rightarrow$

$$-\frac{1}{y} = -\sin x + C \Rightarrow$$

$$\frac{1}{y} = \sin x - C \Rightarrow$$

$$\boxed{y(x) = \frac{1}{\sin x - C}}$$