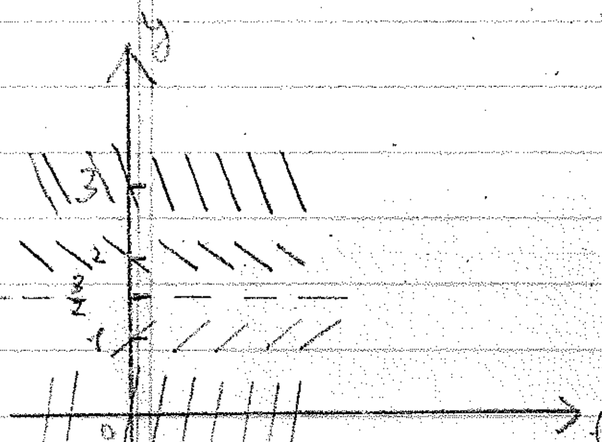
Homework #2 – Solutions MATH308 Fall 2012

Problem 1 $y' = 3 - 2y$

(a) Equilibrium solutions $3-2y=0 \Leftrightarrow y = \frac{3}{2}$

(b)

y	$\frac{3}{2}$	1	2	0	3
slope	0	1	-1	3	-3



$\Downarrow$

the slope is negative if $y > \frac{3}{2}$ and positive if $y < \frac{3}{2}$

Based on the direction field

(c) For any y_0 $\lim\limits_{t\to+\infty} y(t) = \frac{3}{2}$

If $y_0 > \frac{3}{2}$ $\lim\limits_{t\to-\infty} y(t) = +\infty$

If $y_0 = \frac{3}{2}$ $\lim\limits_{t\to-\infty} y(t) = \frac{3}{2}$

If $y_0 < \frac{3}{2}$ $\lim\limits_{t\to-\infty} y(t) = -\infty$

(d) According to the formula we discussed in class

$$y(t) = \frac{3}{2} + \left(y_0 - \frac{3}{2}\right)e^{-2t}$$

From the explicit form

$$\lim_{t \to +\infty} y(t) = \frac{3}{2} \quad \text{for any } y_0$$

$$\lim_{t \to -\infty} y(t) = \begin{cases} +\infty & y_0 > \frac{3}{2} \\ \frac{3}{2} & y_0 = \frac{3}{2} \\ -\infty & y_0 < \frac{3}{2} \end{cases}$$

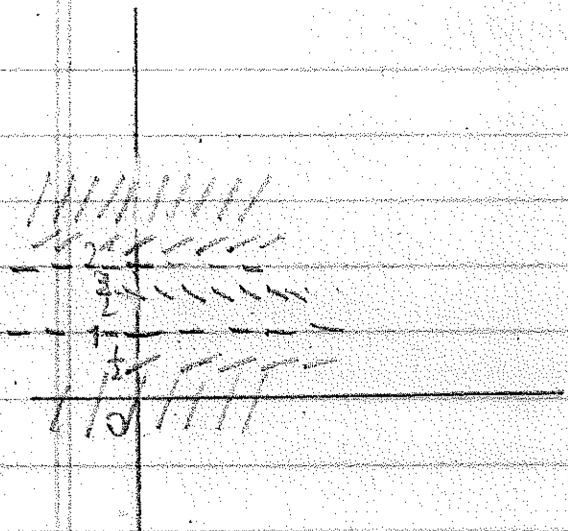
<u>Problem 2</u>

(a) Equilibrium points: $y^2 - 3y + 2 = 0 \Leftrightarrow$
$(y-1)(y-2) = 0 \Rightarrow$

$y = \boxed{1}$ or $y = \boxed{2}$

(b)

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
slope	2	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	2



The slope is positive for $y > 2$ or $y < 1$

The slope is negative for $1 < y < 2$

(c) (i) $y(0) = \frac{4}{3}$, $1 < \frac{4}{3} < 2 \Rightarrow$ (from uniqueness theorem)

$1 < y(t) < 2$ for any $t \Rightarrow y'(t) < 0 \Rightarrow y(t)$ is monotonically decreasing $\Rightarrow$

$$\lim_{t \to +\infty} y(t) = \boxed{1}$$
$$\lim_{t \to -\infty} y(t) = \boxed{2}$$

(Because these limits exist and must be equal to an equilibrium point)

(ii) Similar analysis shows that

$$\lim_{t \to +\infty} y(t) = 1 \text{ for all } y_0 < 2$$

(iii) $y(0) = 3 > 2 \Rightarrow$ from uniqueness $y(t) > 2$ for any $t \Rightarrow y'(t) > 0$. Besides the function $f(y) = y^2 - 3y + 2$ is increasing on $y > 2 \Rightarrow$

$$y'(t) = y^2(t) - 3y(t) + 2 > y_0^2 - 3y_0 + 2 > 0 \Rightarrow$$

$y(t) \geq (y_0^2 - 3y_0 + 2)t + y_0 \xrightarrow[t \to +\infty]{} +\infty$

$\Rightarrow y(t)$ increases to $+\infty$ as t increases.

(d) $y' = y^2 - 3y + 2 \Rightarrow$

$\frac{dy}{y^2-3y+2} = dt \Rightarrow \int \frac{dy}{y^2-3y+2} = \int dt + C_1 = t + C_1$

$\frac{1}{y^2-3y+2} = \frac{1}{(y-1)(y-2)} = \frac{1}{y-2} - \frac{1}{y-1} \Rightarrow \int \left(\frac{1}{y-2} - \frac{1}{y-1}\right) dy = \ln \frac{y-2}{y-1}$

$\ln \frac{y-2}{y-1} = t + C_1 \Rightarrow \frac{y-2}{y-1} = Ce^t$

If $t=0$ then $y=3 \Rightarrow \frac{3-2}{3-1} = C \Rightarrow C = \frac{1}{2} \Rightarrow$

$\frac{y-2}{y-1} = \frac{1}{2}e^t \Rightarrow y - 2 = \frac{1}{2}e^t(y-1) \Rightarrow$

$y\left(1 - \frac{e^t}{2}\right) = 2 - \frac{e^t}{2} \Rightarrow$

$y(t) = \frac{2 - \frac{e^t}{2}}{1 - \frac{e^t}{2}} = \frac{4-e^t}{2-e^t} \Rightarrow$

$2 - e^t = 0 \Rightarrow t = \ln 2 \Rightarrow \lim_{t \to \ln 2^-} y(t) = +\infty$

Since the initial time is $t_0 = 0 \Rightarrow$ the solution is defined for $\boxed{t < \ln 2}$