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Solution of homework #2 MATH 308

Problem 1 (a) $y' = -y\cot t + \cos t$

$y\left(\frac{\pi}{2}\right) = 4$

Solution

$y' + \cot t \, y = \cos t$

$p(t) = \cot t, \quad g(t) = \cos t$

$\Rightarrow$ The integrating factor satisfies

$\mu' = \cot t \, \mu \Rightarrow \mu = c e^{\int \cot t \, dt} = c e^{\ln|\sin t|} =$

$= c|\sin t|$

In fact, we can take $\mu = \sin t$ (adjusting constant C on intervals where $\sin t$ has different signs)

$(\sin t \, y)' = \sin t \cos t = \frac{1}{2}\sin 2t \Rightarrow$

$\sin t \, y = \frac{1}{2}\int \sin 2t \, dt + C = -\frac{1}{4}\cos 2t + C \Rightarrow$

general solution is $y(t) = -\frac{1}{4}\frac{\cos 2t}{\sin t} + \frac{C}{\sin t}$ or, equivalently,

$y(t) = -\frac{1}{4}\frac{2\cos^2 t - 1}{\sin t} + \frac{C}{\sin t} = -\frac{1}{2}\cos t \cot t + \overbrace{\left(C + \frac{1}{4}\right)}^{\tilde{C}}\frac{1}{\sin t}$

$$y\left(\tfrac{\pi}{2}\right) = -\frac{1}{4}\frac{\cos\pi}{\underbrace{\sin\frac{\pi}{2}}_{1}} + \frac{C}{\underbrace{\sin\frac{\pi}{2}}_{1}} = -\frac{1}{4} + C = 4 \Rightarrow C = 4 - \frac{1}{4} = 15$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{4}\frac{\cos 2t}{\sin t} + \frac{15}{4}\frac{1}{\sin t}}$$

or equivalently, $\boxed{y(t) = -\frac{1}{2}\frac{\cos^2 t}{\sin t} + 4\frac{1}{\sin t}}$

Problem 1 (b)

$$(1+t^2)y' + ty = (1+t^2)^{5/2}$$

Solution Divide by $(1+t^2)$:

$$y' + \frac{t}{1+t^2}y = (1+t^2)^{3/2}$$

We can take

$$P(t) = \frac{t}{1+t^2} \Rightarrow \mu' = \frac{t}{1+t^2}\mu \Rightarrow \mu = e^{\int \frac{t}{1+t^2}dt} = e^{\frac{1}{2}\ln(1+t^2)} = (1+t^2)^{1/2}$$

$$\Rightarrow \left((1+t^2)^{1/2}y\right)' = \underbrace{(1+t^2)^{3/2}}_{g(t)} \cdot \underbrace{(1+t^2)^{1/2}}_{\mu(t)} = (1+t^2)^2 = 1 + 2t^2 + t^4 \Rightarrow$$

$$(1+t^2)^{1/2}y(t) = t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C \Rightarrow \boxed{y(t) = \frac{1}{(1+t^2)^{1/2}}\left(t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C\right)}$$

Problem 2 $\quad y' + ay = be^{-ct}$

$P(t) = a, \; g(t) = be^{-ct}$

$\mu' = a\mu$ and μ can be taken as $\mu = e^{at} \Rightarrow$

$$\Rightarrow \left(e^{at}y\right)' = \underbrace{be^{-ct}}_{g(t)}\underbrace{e^{at}}_{\mu(t)} = be^{(a-c)t}$$

$$e^{at}y(t) = b\int e^{(a-c)t}dt + \underbrace{A}_{\text{constant}}$$

(I use this letter, ~~because we already use c for something else~~)

We need to consider two cases here

Case 1 $c \neq a$ then

$$e^{at}y(t) = \frac{b}{a-c}e^{(a-c)t} + A \Rightarrow$$

$$y(t) = \frac{b}{a-c}e^{-ct} + Ae^{-at} \xrightarrow[t\to+\infty]{} 0$$

independently of A, because $a, c > 0$

Case 2 $c = a$

$$e^{at}y(t) = b\int dt + A = bt + A$$

$$y(t) = (bt + A)e^{-at} \xrightarrow[t\to+\infty]{} 0$$

again independently of A

Problem 3

$$y' = y^2 - 2y - 3$$

(a) Equillibrium points : $y^2 - 2y - 3 = 0$

$$D = 4 + 4 \cdot 3 = 16$$

$$y_1 = \frac{2+\sqrt{16}}{2} = \frac{2+4}{2} = 3$$

$$y_2 = \frac{2-\sqrt{16}}{2} = \frac{2-4}{2} = -1$$

So there are two equillibrium points $\boxed{y_1 = 3, y_2 = -1}$

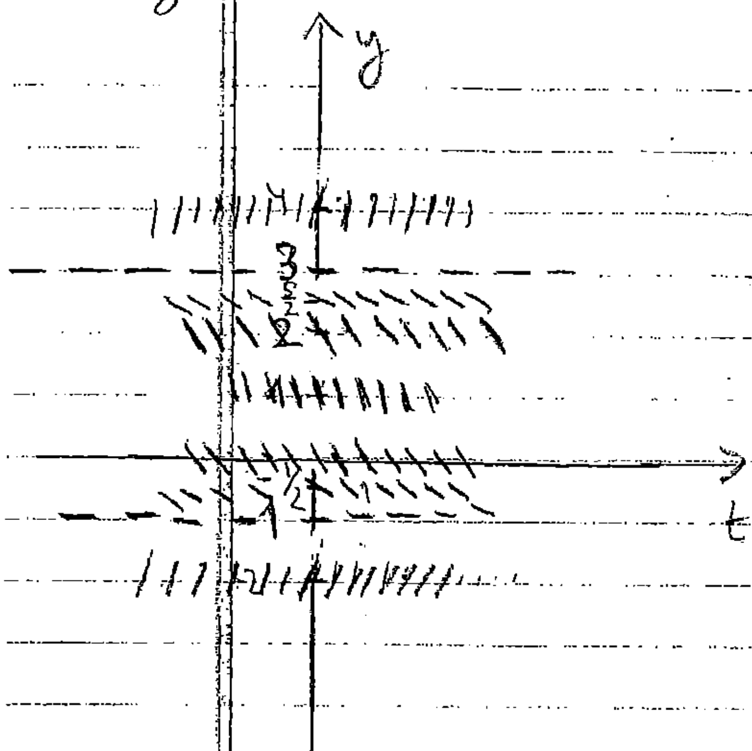
(b) Before sketching the direction field, note that since $y^2 - 2y - 3 = (y-3)(y+1)$ then

$+ \quad -1 \quad - \quad 3 \quad + \quad \to y$

the slope is positive for $y < -1$ or $y > 3$ and it is negative for $-1 < y < 3$

y	-2	-1	0	1	2	3	4	$-\frac{1}{2}$	$\frac{5}{2}$
slope	5	0	-3	-4	-3	0	5	$-\frac{7}{4}$	$-\frac{7}{4}$

See the sketch on the next page



(c) (i) $-1<2<3$ Since $y_1(t)=3$ and $y_2(t)\equiv -1$ are the solution (the equilibrium solutions) from the uniqueness Theorem if $-1<y(0)<3$, then $-1<y(t)<3$ for any t $\Rightarrow$

Since $y^2-2y-3=(y+1)(y-3)<0$ on the interval $-1<y<3$

then $y'(t)=(y(t)+1)(y(t)-3)<0 \Rightarrow y(t)$ is monotonically decreasing and also $\boxed{\lim_{t\to+\infty} y(t)=-1}$ and $\boxed{\lim_{t\to-\infty} y(t)=3}$

(More explanation: As discussed in class, since $y(t)$ is monotonically decreasing and bounded by -1 and 3, both $\lim_{t\to+\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$ exist. Besides, the limits must be equilibrium points, which imply our conclusion).

c(ii) Analyze, what is $\lim_{t\to-\infty} y(t)$ for different initial conditions y_0

(1) If $-1 < y_0 < 3$ then by exactly the same arguments as in c(i) $\lim_{t\to-\infty} y(t) = 3$, i.e. our answer will contain the interval $-1 < y_0 < 3$

(2) $y_0 = 3 \Rightarrow y(t) = 3$ (because $y_0 = 3$ is an equilibrium point) $\Rightarrow \lim_{t\to-\infty} y(t) = 3 \Rightarrow$ the answer will include $y_0 = 3$

(3) $y_0 > 3$. ~~Then from uniqueness $y(t) > 3$ for any t~~ ~~(because the graph of $y(t)$ cannot intersect the graph~~ of the equillibrium solution)

Since $(y+1)(y-3) > 0$ for $y > 3$, $y'(t) > 0$ for every t
$\Rightarrow$ $y(t)$ is monotonically increasing, so $y(t)$ is decreasing as t is decreasing $\Rightarrow \lim_{t\to-\infty} y(t) = 3 \Rightarrow$ the answer will contain $y_0 > 3$

(4) $y_0 = -1 \Rightarrow y(t) \equiv -1$ (because $y_0 = -1$ is an equilibrium point) $\Rightarrow \lim\limits_{t\to\infty} y(t) = -1 \Rightarrow y_0 = -1$ is not in the answer

(5) $y_0 < -1$. Then from uniqueness $y(t) < -1$ for any t (because the graph of $y(t)$ cannot intersect the graph of the equilibrium solution $y_2(t) \equiv -1$) $\Rightarrow$

$\lim\limits_{t\to\infty} y(t)$ cannot be equal to $3 \Rightarrow y_0 < -1$ is not included in the answer

Combining the conclusions of all 5 cases we get that the answer is $\boxed{y_0 > -1}$

c(ii) Since as explained in c(i) case (3)

for $y_0 = 4 > 3$ $y(t)$ is monotonically increasing and $y(t) > 3$ for all t. $f(y) = (y+1)(y-3)$ is monotonically increasing

for $y > 3 \Rightarrow$ for all $t > 0$

$$y'(t) = f(y(t)) > f(y_0) = 5 \Rightarrow$$

$y(t) > y_0 + 5t \Rightarrow$ as t increases $y(t)$ tends to $\boxed{+\infty}$

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(d) $y' = y^2 - 2y - 3 = (y+1)(y-3)$

$$\frac{dy}{(y+1)(y-3)} = dt \Rightarrow \int \frac{dy}{(y+1)(y-3)} = t + C$$

Partial fraction decomposition

$$\frac{1}{(y+1)(y-3)} = \frac{A}{y+1} + \frac{B}{y-3} = \frac{1}{4}\left(\frac{1}{y-3} - \frac{1}{y+1}\right)$$

$$1 = A(y-3) + B(y+1)$$

Plug $y = 3 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

Plug $y = -1 \Rightarrow 1 = -4A \Rightarrow A = -\frac{1}{4}$

$$\Rightarrow \frac{1}{4}\left(\ln|y-3| - \frac{1}{4}\ln|y+1|\right) = t + C_1$$

$$\ln\left|\frac{y-3}{y+1}\right| = 4t + \underbrace{4C_1}_{C_2} \Rightarrow$$

$$\frac{y-3}{y+1} = \underbrace{\pm e^{C_2}}_{C} e^{4t} = Ce^{4t}$$

If $y(0) = 4 \Rightarrow \frac{4-3}{4+1} = C \Rightarrow C = \frac{1}{5} \Rightarrow$

$$\frac{y-3}{y+1} = \frac{1}{5}e^{4t} \Rightarrow y - 3 = \frac{1}{5}e^{4t}y + \frac{1}{5}e^{4t} \Rightarrow$$

$$y\left(1 - \frac{1}{5}e^{4t}\right) = 3 + \frac{1}{5}e^{4t} \Rightarrow y = \frac{3 + \frac{1}{5}e^{4t}}{1 - \frac{1}{5}e^{4t}} = \frac{15 + e^{4t}}{5 - e^{4t}}$$

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So, $y(t) = \frac{15+e^{4t}}{5-e^{4t}}$ $\Rightarrow$ the point of discontinuity (blow up) is when $5-e^{4t}=0 \Leftrightarrow t = \frac{1}{4}\ln 5$

So the solution of IVP is defined for $\boxed{t<\frac{1}{4}\ln 5}$.

Problem 4 (a)

$$y' = \frac{y-x}{y+x} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1}$$

$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = xu' + u$

$$xu'+u = \frac{u-1}{u+1} \Rightarrow xu' = \frac{u-1}{u+1} - u = \frac{u-1-u^2-u}{u+1} \Rightarrow$$

$$xu' = -\frac{u^2+1}{u+1} \Rightarrow \text{separable}$$

$$\frac{u+1}{u^2+1}du = -\frac{1}{x}dx \Rightarrow \int\frac{u+1}{u^2+1}du = -\ln|x|+C$$

$$\int\frac{u+1}{u^2+1}du = \underbrace{\int\frac{u}{u^2+1}du}_{\frac{1}{2}\ln(u^2+1)} + \underbrace{\int\frac{du}{u^2+1}}_{\arctan(u)} = -\ln|x|+C$$

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$\Rightarrow \frac{1}{2}\ln(u^2+1) + \arctan(u) = -\ln|x| + C$

$\Rightarrow$ since $u = \frac{y}{x}$ we get

$\boxed{\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+1\right) + \arctan\left(\frac{y}{x}\right) = -\ln|x| + C} \Leftrightarrow$

cancelled

$\frac{1}{2}\ln(x^2+y^2) - \ln|x|$

Problem 4 (b)

$\frac{1}{2}\ln(x^2+y^2) + \arctan\left(\frac{y}{x}\right) = C$

$y' = (3x+2y-1)^2$

Let $u(x) = 3x+2y(x)-1 \Rightarrow u'(x) = 3+2y'(x) \Rightarrow$

$y'(x) = \frac{1}{2}u'(x) - \frac{3}{2}$

$\frac{1}{2}u'(x) - \frac{3}{2} = u^2 \Rightarrow \quad u' = 2u^2+3 \rightarrow$ separable

$\Rightarrow \frac{du}{2u^2+3} = dx \Rightarrow \int \frac{du}{2u^2+3} = x + C$

$\int \frac{du}{2u^2+3} = \frac{1}{3}\int \frac{du}{\frac{2}{3}u^2+1} = \frac{1}{3}\sqrt{\frac{3}{2}}\int \frac{dz}{z^2+1} = \frac{1}{\sqrt{6}}\arctan z =$

$z = \sqrt{\frac{2}{3}}u \quad dz = \sqrt{\frac{2}{3}}du \Rightarrow$

$du = \sqrt{\frac{3}{2}}dz$

$= \frac{1}{\sqrt{6}}\arctan\sqrt{\frac{2}{3}}u \Rightarrow \quad \frac{1}{\sqrt{6}}\arctan\sqrt{\frac{2}{3}}u = x + C_1$

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$u = 3x+2y-1$

$$\arctan \sqrt{\tfrac{2}{3}}\,(3x+2y-1) = \sqrt{6}\,x + \underbrace{\sqrt{6}\,C_1}_{C}$$

$$\sqrt{\tfrac{2}{3}}\,(3x+2y-1) = \tan(\sqrt{6}\,x + C) \Rightarrow$$

$$3x+2y-1 = \sqrt{\tfrac{3}{2}}\,\tan(\sqrt{6}\,x+C) \Rightarrow$$

$$2y = \sqrt{\tfrac{3}{2}}\,\tan(\sqrt{6}\,x+C) + 1 - 3x$$

$$\boxed{y = \frac{1}{2}\sqrt{\frac{3}{2}}\,\tan(\sqrt{6}\,x+C) + \frac{1}{2} - \frac{3x}{2}}$$