Solutions of homework assignment #2 MATH 308 - Summer 2012

Problem 1

$$y' - xy^2 = xy \quad , y(0) = 1$$

$$y' = xy^2 + xy = x(y^2+y)$$

$$\frac{y'}{y^2+y} = x \iff \frac{dy}{y(y+1)} = x\,dx$$

$$\int \frac{dy}{y(y+1)} = \int x\,dx + C_1$$

$$\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

$$\ln\left|\frac{y}{y+1}\right| = \frac{x^2}{2} + C_1 \Rightarrow$$

$$\left|\frac{y}{y+1}\right| = e^{\frac{x^2}{2}+C_1} = \underbrace{e^{C_1}}_{C} e^{\frac{x^2}{2}} = Ce^{\frac{x^2}{2}} \Rightarrow$$

$$\frac{y}{y+1} = Ce^{\frac{x^2}{2}} \Rightarrow y = Ce^{\frac{x^2}{2}}(y+1) \Rightarrow$$

$$y\left(1 - Ce^{\frac{x^2}{2}}\right) = Ce^{\frac{x^2}{2}} \Rightarrow$$

$$y = \frac{Ce^{\frac{x^2}{2}}}{1 - Ce^{\frac{x^2}{2}}}$$

$$y(0) = 1 \Rightarrow \quad 1 = \frac{Ce^0}{1-Ce^0} = \frac{C}{1-C} \Rightarrow 1 - C = C \Rightarrow C = \frac{1}{2} \Rightarrow$$

$$\boxed{y = \frac{\frac{1}{2}e^{\frac{x^2}{2}}}{1 - \frac{1}{2}e^{\frac{x^2}{2}}}}$$

Problem 2

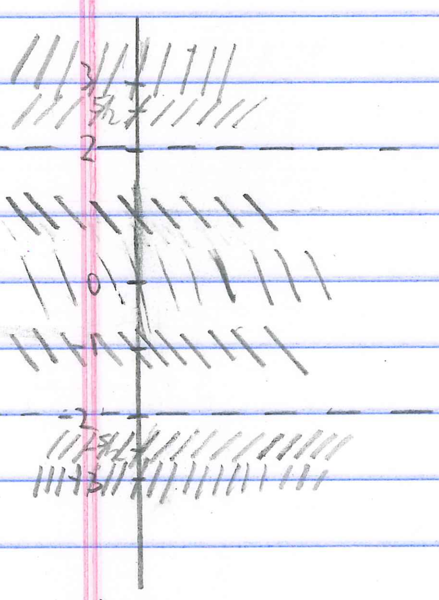
(a) Equilibrium points satisfy

$$y^2-4=0 \iff \boxed{y=2 \text{ or } y=-2}$$

(b) $y^2-4>0 \iff y>2$ or $y<-2$

$y^2-4<0 \iff -2<y<2$

y	-3	$-\frac{5}{2}$	-2	-1	0	1	2	$\frac{5}{2}$	3
slope	5	$\frac{9}{4}$	0	-3	-4	3	0	$\frac{9}{4}$	5



(c) (i) Since 2 and -2 are equilibrium points then $y_1(t)\equiv 2$ and $y_2(t)\equiv -2$ are solutions of our equation. From this and the uniqueness of solution of IVP it follows that if $y(0)=-\frac{1}{3}$ then $-2<y(t)<2 \Rightarrow y'(t)<0 \Rightarrow$ $y(t)$ is monotonically decreasing (and also bounded) $\Rightarrow$ $\lim\limits_{t\to+\infty} y(t)$ exists. Moreover if $\lim\limits_{t\to+\infty} y(t)=C$ then C must be a stationary point $\Rightarrow C=-2$, i.e. $\lim\limits_{t\to+\infty} y(t)=-2$

In the same way $\lim_{t \to -\infty} y(t) = 2$

ii) By the same arguments if $y(0) = y_0$ with

$-2 < y_0 < 2$, then $\lim_{t \to +\infty} y(t) = -2$

Further, if $y(0) = y_0 < -2$ then $y(t) < -2 \Rightarrow$

$\Rightarrow y'(t) > 0 \Rightarrow y(t)$ is monotonically increasing

$\Rightarrow \lim_{t \to +\infty} y(t) = c \leq -2$ and c must be a stationary point $\Rightarrow$

$c = -2 \Rightarrow \lim_{t \to +\infty} y(t) = -2$

Finally if $y(0) = y_0$ with $y_0 \geq 2$

then $y(t) \geq 2 \Rightarrow y'(t) \geq 0$ for any $t \Rightarrow$

$y(t)$ is increasing $\Rightarrow y(t)$ does not converge to

-2 as $t \to +\infty \Rightarrow$ the answer to item c ii

is $\boxed{y_0 < 2}$

iii) As mentioned in the previous item if $y(0) = y_0$

and $y_0 > 2$ then $y(t)$ is increasing. Moreover, it

converges to $+\infty$ (and in finite time)

It converges to infinity because $y'(t) \geq y_0^2 - 4 > 0 \Rightarrow$

$$y(t) > (y_0^2 - 4)t + y_0 \xrightarrow[t \to +\infty]{} +\infty$$

The fact that it goes to infinity in finite time follows basically from the fact that $y^2-4 \sim y^2$ as $y \to +\infty$.

(for explanation via explicit solution see item d)

d) Let us find the solution $y(t)$ with $y(0)=3$ explicitly:

$$y' = y^2-4 \iff \frac{dy}{y^2-4} = dt$$

$$\int \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = t + c_1$$

$$\frac{1}{4} \ln\left|\frac{y-2}{y+2}\right| = t + c_1 \iff \ln\left|\frac{y-2}{y+2}\right| = 4t + \underbrace{4c_1}_{c_2}$$

$$\frac{y-2}{y+2} = \underbrace{e^{4c_1}}_{C} e^{4t} = Ce^{4t}$$

$$y(0)=3 \Rightarrow \frac{1}{5} = Ce^0 \Rightarrow C = \frac{1}{5} \Rightarrow$$

$$\frac{y-2}{y+2} = \frac{1}{5}e^{4t} \Rightarrow$$

$$y-2 = \frac{1}{5}e^{4t}(y+2) \Rightarrow$$

$$y\left(1-\frac{1}{5}e^{4t}\right) = 2 + \frac{2}{5}e^{4t} \Rightarrow$$

$$y(t) = \boxed{\frac{2+\frac{2}{5}e^{4t}}{1-\frac{1}{5}e^{4t}}} \Rightarrow 1-\frac{1}{5}e^{4t} \neq 0 \iff e^{4t} \neq 5 \iff$$

$$t \neq \frac{1}{4}\ln 5$$

Since $t_0 = 0$ then the solution is defined for $t < \frac{1}{4}\ln 5$. When $t \to \frac{1}{4}\ln 5^-$ the solution "exploses" to $+\infty$.