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Homework Assignment #3 solutions MATH308

Problem 1

(a) $$\frac{dQ}{dt} = \underbrace{r\gamma(t)}_{\text{flow in}} - \underbrace{\frac{r}{V}Q(t)}_{\text{flow out}}$$

$V = 300$

$\gamma(t) = 0.5 - 0.2\sin 2t \Rightarrow r\gamma(t) = 2.5 - \sin$

$r = 5$ $\qquad \frac{r}{V} = \frac{1}{60}$

$\Downarrow$

The differential equation is

$$\frac{dQ}{dt} = 2.5 - \sin 2t - \frac{1}{60}Q$$

initial condition $\quad Q(0) = 40$

(b) We have a linear equation. Solve it by the method of integrating factor

$$\frac{dQ}{dt} + \frac{1}{60}Q = 2.5 - \sin 2t$$

$\Rightarrow \mu' = \frac{1}{60}\mu \Rightarrow$ we can take $\mu = e^{\frac{t}{60}} \Rightarrow$

$$\left(e^{\frac{t}{60}}Q\right)' = 2.5e^{\frac{t}{60}} - e^{\frac{t}{60}}\sin 2t \Rightarrow$$

$$e^{\frac{t}{60}}Q = 2.5\underbrace{\int e^{\frac{t}{60}}dt}_{\frac{e^{\frac{t}{60}}}{\frac{1}{60}}} - \int e^{\frac{t}{60}}\sin 2t\,dt + C$$

$$\underbrace{2.5\,\frac{e^{\frac{t}{60}}}{\frac{1}{60}}}_{60\cdot 2.5\,e^{\frac{t}{60}} = 150e^{\frac{t}{60}}}$$

$$\underbrace{\int e^{\frac{t}{60}}\sin 2t\,dt}_{\begin{array}{l} u=e^{\frac{t}{60}} \quad v'=\sin 2t \\ u'=\frac{1}{60}e^{\frac{t}{60}} \quad v=-\frac{1}{2}\cos 2t\end{array}} \rightarrow I = -\frac{1}{2}e^{\frac{t}{60}}\cos 2t + \frac{1}{120}\underbrace{\int e^{\frac{t}{60}}\cos 2t\,dt}_{\begin{array}{l} u=e^{\frac{t}{60}} \quad v'=\cos 2t \\ u'=\frac{1}{60} \quad v=\frac{1}{2}\sin 2t\end{array}} =$$

$$= -\frac{1}{2}e^{-\frac{t}{60}}\cos 2t + \frac{1}{240}e^{\frac{t}{60}}\sin 2t - \frac{1}{14400}\int e^{\frac{t}{60}}\sin 2t\,dt$$

$$\left(1+\frac{1}{14400}\right)\int e^{\frac{t}{60}}\sin 2t\,dt = -\frac{1}{2}e^{\frac{t}{60}}\cos 2t + \frac{1}{240}e^{\frac{t}{60}}\sin 2t$$

$$\int e^{\frac{t}{60}}\sin 2t\,dt = \frac{14400}{14401}\left(-\frac{1}{2}e^{\frac{t}{60}}\cos 2t + \frac{1}{240}e^{\frac{t}{60}}\sin 2t\right) =$$

$$= -\frac{7200}{14401}e^{\frac{t}{60}}\cos 2t + \frac{60}{14401}e^{\frac{t}{60}}\sin 2t$$

$$\Downarrow$$

$$e^{\frac{t}{60}}Q = 150e^{\frac{t}{60}} + \frac{7200}{14401}e^{\frac{t}{60}}\cos 2t - \frac{60}{14401}e^{\frac{t}{60}}\sin 2t + C \Rightarrow$$

$$Q(t) = 150 + \frac{7200}{14401}\cos 2t - \frac{60}{3601}\sin 2t + Ce^{-\frac{t}{60}}$$

$Q(0)=40$ $$40 = 150 + \frac{720}{14401} + C \Rightarrow \boxed{C = -\frac{720}{14401} - 110}$$

$$\boxed{Q(t) = 150 + \frac{720}{14401}\cos 2t - \frac{60}{3601}\sin 2t - \left(\frac{720}{14401}+110\right)e^{-\frac{t}{60}}}$$

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Problem 2 $(4t^2+5t-6)y' + \ln|2t^2+9t-5|\,y = e^{t^3}+\sin(t^2)$

II

Divide by $4t^2+5t-6$

$$y' + \underbrace{\frac{\ln|2t^2+9t-5|}{4t^2+5t-6}}_{p(t)}\,y = \underset{q(t)}{\frac{e^{t^3}+\sin(t^2)}{4t^2+5t-6}}$$

1) Find the points of discontinuity of $P(t)$:

Either $4t^2+5t-6=0$ or $2t^2+9t-5=0$

$D=5^2-4\cdot4\cdot(-6)=25+96=121$ $D=9^2-4\cdot2\cdot(-5)=81+40=121$

$t_1=\frac{-5+11}{8}=\frac{6}{8}=\frac{3}{4}$ $t_3=\frac{-9+11}{4}=\frac{1}{2}$

$t_2=\frac{-5-11}{8}=-2$ $t_4=\frac{-9-11}{4}=-5$

So P has 4 points of discontinuity:

$-5, -2, \frac{1}{2}, \frac{3}{4}$

2) Find the points of discontinuity of $q(t)$: $\rightarrow$ only when $4t^2+5t-6=0$, i.e $\frac{3}{4}$ and -2 that already appeared in the previous item.

So dodally the points of discontinuity of P or q are $\{-5, -2, \frac{1}{2}, \frac{3}{4}\}$

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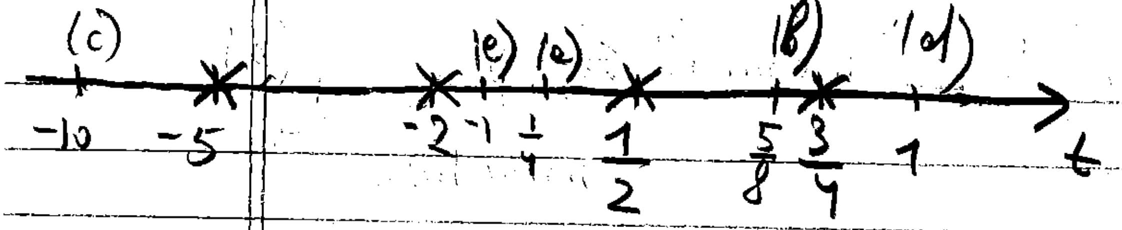

The answers can be deduced from the pictures above

(a) $t_0 = \frac{1}{4}$ lies in the interval $(-2, \frac{1}{2})$ and this is the maximal interval where both $P(t)$ and $g(t)$ are continues $\Rightarrow$ $\boxed{-2 < t < \frac{1}{2}}$

(b) $t_0 = 5/8$. By the same arguments $\boxed{\frac{1}{2} < t < \frac{3}{4}}$

(c) $t_0 = -10$. By the same arguments $\boxed{t < -5}$

(d) $t_0 = 1$ By the same arguments $\boxed{t > \frac{3}{4}}$

(e) $t_0 = -1$ By the same arguments $\boxed{-2 < t < \frac{1}{2}}$

Problem 3

$$\underbrace{y\sec^2 x + \sec x \tan x}_{P} + \underbrace{(2y + \tan x)}_{Q}\frac{dy}{dx}$$

$P = y\sec^2 x + \sec x \tan x \Rightarrow P_y = \sec^2 x$

$Q = 2y + \tan x \Rightarrow Q_x = \sec^2 x$

$\Rightarrow P_y = Q_x \Rightarrow$ it is exact

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$\varphi_x = y\sec^3 x + \sec x \tan x$ (Eq 1)

(Eq 2)

$\varphi_y = 2y + \tan x \Rightarrow \varphi = \int (2y + \tan x)\,dy + h(x) =$

$= y^2 + \tan x\, y + h'(x)$

$\varphi_x = \cancel{\sec^3 x\, y} + h'(x) = \cancel{y \sec^2 x} + \sec x \tan x$

$h'(x) = \underbrace{\sec x \tan x}_{\frac{\sin x}{\cos^2 x}} \Rightarrow h(x) = \int \frac{\sin x\, dx}{\cos^2 x} = -\int \frac{du}{u^2} =$

$u = \cos x$

$du = -\sin x\, dx$

$= \frac{1}{u} = \frac{1}{\cos x} + C$

$\Rightarrow$ we can take as a potential

$\varphi = y^2 + \tan x\, y + \frac{1}{\cos x}$ (i.e. with $C = 0$)

general solution is described by the level set of φ:

$y^2 + \tan x\, y + \frac{1}{\cos x} = C$

(more precisely the graphs of solutions lie on the level sets). $y(0) = -2$; $4 + 1 = C \Rightarrow C = 5 \Rightarrow$

$\boxed{y^2 + \tan x\, y + \frac{1}{\cos x} = 5}$

Problem 4 $\left(e^{ax+y} + 3x^2y^2\right)dx = -\left(2yx^3 + e^{ax+y}\right)dy$

$\left(\underbrace{e^{ax+y} + 3x^2y^2}_{P}\right)dx + \left(\underbrace{2yx^3 + e^{ax+y}}_{Q}\right)dy$

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$P = e^{ax+y} + 3x^2y^2 \Rightarrow P_y = e^{ax+y} + 6x^2y$

$Q = 2yx^3 + e^{ax+y} \Rightarrow Q_x = 6yx^2 + ae^{ax+y}$

$$P_y = Q_x \Leftrightarrow e^{ax+y} + \cancel{6x^2y} = \cancel{6yx^2} + ae^{ax+y} \Leftrightarrow$$

$$\cancel{e^{ax+y}} = a\cancel{e^{ax+y}} \Leftrightarrow \underline{a = 1}$$

So the equation is exact $\Leftrightarrow \boxed{a = 1}$

Solve the equation for $a = 1$

$\varphi_x = e^{x+y} + 3x^2y^2 \Leftrightarrow \varphi = \int(e^{x+y} + 3x^2y^2)dx + h(y)$

$\varphi_y = 2yx^3 + e^{x+y}$ $\qquad = e^{x+y} + x^3y^2 + h(y)$

$$\varphi_y = \cancel{e^{x+y}} + \cancel{2yx^3} + h'(y) = \cancel{2yx^3} + \cancel{e^{x+y}} \Rightarrow$$

$h'(y) = 0 \Rightarrow$ we can define $h(y) = 0$ and

$\varphi = e^{x+y} + x^3y^2 \Rightarrow$ the general solution

is $\boxed{e^{x+y} + x^3y^2 = C}$

Problem 5 $\qquad y^2\sin x + yf(x)\frac{dy}{dx} = 0$

$P = y^2\sin x \Rightarrow P_y = 2y\sin x \Rightarrow \quad P_y = Q_x \Leftrightarrow 2y\sin x = yf'(x)$

$Q = yf(x) \Rightarrow Q_x = yf'(x) \qquad \Rightarrow f'(x) = 2\sin x$

$\Rightarrow f(x) = 2\int \sin x + \underbrace{A}_{\text{a constant}} = -2\cos x + A$

$\Rightarrow \boxed{f(x) = -2\cos x + A}$, where A is an arbitrary constant

Solve the equation for such $f(x)$,

$$y^2 \sin x + y(-2\cos x + A)\frac{dy}{dx} = 0$$

Find the potential

$\varphi_x = y^2 \sin x \Rightarrow \varphi = \int y^2 \sin x \, dx + h(y) = -y^2\cos x + h(y) \Rightarrow$

$\varphi_y = y(-2\cos x + A) = -2y\cos x + Ay$

$\varphi_y = \cancel{-2y\cos x} + h'(y) = \cancel{-2y\cos x} + Ay \Rightarrow$

$h'(y) = Ay \Rightarrow$ We can define $h(y) = \frac{A}{2}y^2 \Rightarrow$

and $\varphi = -y^2\cos x + \frac{A}{2}y^2 \Rightarrow$ the general solution is

$\boxed{-y^2\cos x + \frac{A}{2}y^2 = C} \Leftrightarrow y^2 = \frac{C}{\frac{A}{2} - \cos x} \Leftrightarrow$

$y = \pm \frac{C}{\sqrt{\frac{A}{2} - \cos x}}$

Problem 6

(a) $$y' + p(t)y = q(t)y^n \quad t>0$$

Divide by y^n : $y'y^{-n} + p(t)y^{1-n} = q(t)$ (+)

If $v = y^{1-n}$ then $\frac{dv}{dx} = (1-n)\frac{dy}{dx}y^{-n} \Rightarrow y'y^{-n} = \frac{1}{1-n}v'$

Substituting in (+) : $\frac{1}{1-n}v' + p(t)v = q(t) \Rightarrow$

a linear equation as required

(b) $$t^2y' + 2ty - y^3 = 0 \Rightarrow$$

$$y' + \frac{2}{t}y = \frac{1}{t^2}y^3 \Rightarrow$$ Bernoulli equation with n=3

Make the substitution $v = y^{1-3} = y^{-2} \Rightarrow v' = -2y^{-3}y'$

$$\underline{y^{-3}y'} + \frac{2}{t}y^{-2} = \frac{1}{t^2}$$

$$-\frac{1}{2}v' + \frac{2}{t}v = \frac{1}{t^2} \Rightarrow$$

$$v' - \frac{4}{t}v = -\frac{2}{t^2}$$

Integrating factor μ satisfies $\mu' = -\frac{4}{t}\mu \Rightarrow$

We can take $\mu = e^{-\int\frac{4}{t}} = e^{-4\ln|t|} = t^{-4}$

$$(t^{-4}v)' = -2t^{-2}\cdot t^{-4} = -2t^{-6} \Rightarrow$$

$$t^{-4}v = -2\int t^{-6}\,dt + C = (-2)\frac{t^{-5}}{(-5)} + C =$$

$$= \frac{2}{5}t^{-5} + C \Rightarrow$$

$$v = \frac{2}{5}t^{-1} + Ct^{4} \Rightarrow$$

Since $v = y^{-2} \Rightarrow y = v^{-1/2} = \pm\frac{1}{\sqrt{\frac{2}{5}t^{-1}+Ct^{4}}} = \boxed{\pm\frac{\sqrt{5}\sqrt{t}}{\sqrt{5Ct^{5}+2}}}$