Homework # 3 - Solutions MATH 308 - Fall 2012

Problem 1

$$y' + 3y = 4e^{-2t} \Rightarrow p = 3,\ g = 4e^{-2t}$$

Integrating factor satisfies:

$$\mu' = 3\mu \Rightarrow \text{one can take } \mu = e^{3t} \Rightarrow$$

$$(e^{3t}y)' = 4e^{-2t}e^{3t} = 4e^{t} \Rightarrow$$

$$e^{3t}y = \int 4e^{t}dt + C = 4e^{t} + C \Rightarrow$$

Gen. solution: $\boxed{y(t) = 4e^{-2t} + Ce^{-3t}} \Rightarrow$

$\boxed{\lim_{t\to+\infty} y(t) = 0}$ for any solution

Problem 2

(a) $$y' - \frac{2y}{t+1} = e^{-3t}(t+1)^2,\ y(0) = a$$

$$p = -\frac{2}{t+1},\ g = e^{-3t}(t+1)^2$$

$$\Downarrow$$

$$\mu' = -\frac{2}{t+1}\mu \Rightarrow \mu = e^{-\int \frac{2}{t+1}dt} = e^{-2\ln|t+1|} = (t+1)^{-2}$$

$$\left((t+1)^{-2}y\right)' = e^{-3t}(t+1)^2(t+1)^{-2} = e^{-3t} \Rightarrow$$

$(t+1)^{-2} y = -\frac{1}{3}e^{-3t} + C \Rightarrow$

$y(t) = (t+1)^2\left(-\frac{1}{3}e^{-3t} + C\right)$

$y(0) = a \Rightarrow \quad a = -\frac{1}{3} + C \Rightarrow C = a + \frac{1}{3}$

$\Downarrow$

$\boxed{y(t) = (t+1)^2\left(-\frac{1}{3}e^{-3t} + a + \frac{1}{3}\right)} = -\frac{1}{3}(t+1)^2 e^{-3t} + \left(a+\frac{1}{3}\right)(t+1)^2$

(b) Everything depends on the sign of $a + \frac{1}{3}$

Note that $-\frac{1}{3}(t+1)^2 e^{-3t} \xrightarrow[t \to +\infty]{} 0$

So if $a + \frac{1}{3} > 0 \Leftrightarrow a > -\frac{1}{3}$ then $y(t) \xrightarrow[t \to +\infty]{} +\infty$

If $a + \frac{1}{3} < 0 \Leftrightarrow a < -\frac{1}{3}$ then $y(t) \xrightarrow[t \to +\infty]{} -\infty$

If $a = -\frac{1}{3}$ then $y(t) = -\frac{1}{3}(t+1)^2 e^{-3t} \xrightarrow[t \to +\infty]{} 0$

The value a_0 for which the transition from one type of behavior occurs is $\boxed{a_0 = -\frac{1}{3}}$

(c) If $a = -\frac{1}{3}$ then $\lim\limits_{t \to +\infty} y(t) = 0$

Problem 3

a) $Q'(t) = \text{rate}_{\text{in}} - \text{rate}_{\text{out}} = 4\left(1+\frac{1}{2}\sin t\right) - \frac{Q(t)}{200}\cdot 4 =$

$= 4 + 2\sin t - \frac{Q(t)}{50} \Rightarrow$

$$\boxed{Q' + \frac{Q}{50} = 4 + 2\sin t \quad , \quad Q(0) = 25\,oz}$$

b) $p = \frac{1}{50}$, $g = 4 + 2\sin t$

$\mu' = \frac{1}{50}\mu \Rightarrow$ one can take $\mu = e^{\frac{t}{50}} \Rightarrow$

$\left(e^{\frac{t}{50}} Q\right)' = (4 + 2\sin t)e^{\frac{t}{50}}$

$e^{\frac{t}{50}} Q(t) = 4\int e^{\frac{t}{50}}dt + 2\int e^{\frac{t}{50}} \sin t\, dt + C$

$= 200e^{\frac{t}{50}} + 2\int e^{\frac{t}{50}} \sin t\, dt + C$

Integration by parts.

$\int \underbrace{e^{\frac{t}{50}}}_{u} \underbrace{\sin t}_{v'}\, dt = -e^{\frac{t}{50}}\cos t + \frac{1}{50}\int \underbrace{e^{\frac{t}{50}}}_{u} \underbrace{\cos t}_{v'} dt =$

$u' = \frac{1}{50}e^{\frac{t}{50}} \mid v = -\cos t$ $\qquad$ $u' = \frac{1}{50}e^{\frac{t}{50}} \mid v = \sin t$

$= -e^{\frac{t}{50}}\cos t + \frac{1}{50}e^{\frac{t}{50}}\sin t - \frac{1}{2500}\int e^{\frac{t}{50}}\sin t\, dt \Rightarrow$

$\left(1 + \frac{1}{2500}\right)\int e^{\frac{t}{50}} \sin t\, dt = -e^{\frac{t}{50}}\cos t + \frac{1}{50}e^{\frac{t}{50}}\sin t \Rightarrow$

$\int e^{\frac{t}{50}}\sin t\, dt = \frac{2500}{2501}\left(-e^{\frac{t}{50}}\cos t + \frac{1}{50}e^{\frac{t}{50}}\sin t\right) =$

$$= -\frac{2500}{2501} e^{\frac{t}{50}} \cos t + \frac{50}{2501} e^{\frac{t}{50}} \sin t$$

$$e^{\frac{t}{50}} Q(t) = 200 e^{\frac{t}{50}} - \frac{5000}{2501} e^{\frac{t}{50}} \cos t + \frac{100}{2501} e^{\frac{t}{50}} \sin t + C \Rightarrow$$

$$Q(t) = 200 - \frac{5000}{2501} \cos t + \frac{100}{2501} \sin t + C e^{-\frac{t}{50}}$$

$$Q(0) = 25 \text{ oz} \Rightarrow \quad 25 = 200 - \frac{5000}{2501} + C \Rightarrow$$

$$C = \frac{5000}{2501} - 175 = 2 - \frac{2}{2501} - 175 =$$

$$= -173 - \frac{2}{2501} = -173\frac{2}{2501} \Rightarrow$$

$$\boxed{Q(t) = 200 - \frac{5000}{2501} \cos t + \frac{100}{2501} \sin t - 173\frac{2}{2501} e^{-\frac{t}{50}}}$$