# Homework Assignment #3 FAll 2016 - MATH308, Honors section

**due February 14, 2017 at the beginning of class**

Topics covered : exactness on non simply connected domain; direction field (sections 1.1) ; existence and uniqueness theorems: difference between linear and nonlinear equations; qualitative analysis of autonomous equations on the line/population dynamics models; phase line, and stability of equilibrium points (corresponds to and 2.5) and a bonus question on Euler' method for numerical approximation of solutions (section 2.7).

1. Consider the differential equation

$$(12t^2 + 5t - 2)y' + \ln|3t^2 - 7t + 2|y = \arctan(t^2 - 4t + 4).$$

In each of the following five items determine (without solving the equation) an interval in which the solution with given initial condition is certain to exist if the initial condition is

(a) $y(-1) = -3$, (b) $y(7/24) = -50$, (c) $y(5) = 20$, (d) $y(1) = -100$, (e) $y(0.2) = -30$.

This problem is based on Theorem 2.4.1 of the book, see also Theorem 4 of week three notes on existence and uniqueness theorem

2. (a) For each item below, based on the conditions of Theorem 2.4.2 of the book (see also Theorem 3 of week three notes on existence and uniqueness theorem) find all points $(t_0, y_0)$ for which there exist a unique solution of the corresponding initial value problem on an interval $(t_0 - h, t_0 + h)$ for some $h > 0$:

   i. $y' = y^{\frac{1}{5}}$, $y(t_0) = y_0$;

   ii. $y' = \dfrac{\ln|ty|}{1 - t^2 + y^2}$, $y(t_0) = y_0$.

   (b) For the differential equation of item (a) i find an example of a point $(t_0, y_0)$ for which the corresponding initial value problem $y(t_0) = y_0$ has more than one solutions. Justify your answer by giving explicit examples of two different solutions of this initial value problem.

3. Given the differential equation:

$$y' = y^2 - 7y + 6 \tag{1}$$

   (a) Find all equilibrium points.

   (b) Sketch a direction field.

   (c) Based on the sketch of the direction field from the item (b) answer the following questions:

      i. Let $y(t)$ be the solution of equation (1) satisfying the initial condition $y(0) = 4$. Find the limit of $y(t)$ when $t \to +\infty$ and the limit of $y(t)$ when $t \to -\infty$ (for this you do not need to find $y(t)$ explicitly).

      ii. Find all $y_0$ such that the solution of the equation (1) with the initial condition $y(0) = y_0$ has the same limit at $+\infty$ as the solution from the item (c)i.

      iii. Let $y(t)$ be the solution of equation (1) with $y(0) = 8$. Decide whether $y(t)$ is monotonically decreasing or increasing and find to what value it approaches when $t$ increases (the value might be infinite).

   (d) Find the solution of the equation (1) with $y(0) = 8$ explicitly. Determine the interval in which this solution is defined.

4. Given the differential equation:

$$y' = (y^2 - 4)(y + 2)(y^2 - y - 12)$$

   (a) Find all equilibrium points;

   (b) Sketch the phase portrait of the equation (4) on the phase line;

   (c) For each equilibrium point determine whether it is asymptotically stable, unstable or semistable.

5. (**The only additional question for the honors section**) Consider the equation

$$\frac{y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy = 0 \tag{2}$$

on $\mathbb{R}^2 \backslash (0, 0)$ (i.e. on the plane $\mathbb{R}^2$ without the origin).

   (a) If $P(x, y) = \dfrac{y}{x^2 + y^2}$ and $Q(x, y) = -\dfrac{x}{x^2 + y^2}$, verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$;

   (b) Prove that the equation (2) is not exact on $\mathbb{R}^2 \backslash (0, 0)$.

6. (**bonus - 25 points**; *you can use a calculator for this problem*) Before attempting this problem review section 2.7 of the book (or the short notes on the Euler method that will be posted soon).

   (a) Using Euler's method with step size $h = 0.2$ determine an approximated value of the solution at $t = 1$ for the initial value problem

$$\frac{dy}{dt} = 1 + t - y, \quad y(0) = 0$$

   and compare the result with the value at $t = 1$ of the actual solution.

   (b) Solve the same problem for the step size $h = 0.1$.