Homework # 3 solution, MATH 308, Regular part

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Problem 1 $\left(12t^2+5t-2\right)y' + \ln|3t^2-7t+2|\,y = \arctan(t^2-4t+4)$

Divide by $12t^2+5t-2$ (i.e. the coefficient of y'):

$$y' + \frac{\ln|3t^2-7t+2|}{12t^2+5t-2}\,y = \frac{\arctan(t^2-4t+4)}{12t^2+5t-2}$$

$$\Rightarrow p(t) = \frac{\ln|3t^2-7t+2|}{12t^2+5t-2}, \quad g(t) = \frac{\arctan(t^2-4t+4)}{12t^2+5t-2}$$

Find all points where either $p(t)$ or $g(t)$ are discontinuous:

1) $p(t)$ is discontinuous as $12t^2+5t+2=0$ or

$3t^2-7t+2=0$.

$12t^2+5t+2=0$

$D = 25+4\cdot 24 = 121$

$t_1 = \frac{-5+11}{24} = \frac{1}{4}$

$t_2 = \frac{-5-11}{24} = -\frac{16}{24} = -\frac{2}{3}$

$3t^2-7t+2=0$

$D = 49-24 = 25$

$t_1 = \frac{7-5}{6} = \frac{1}{3}$

$t_2 = \frac{7+5}{6} = 2$

So there are 4 points of discontinuity: $-\frac{2}{3}, \frac{1}{4}, \frac{1}{3}$ & 2

2) $g(t)$ is discontinuous as $12t^2+5t-2 \Rightarrow$ the points of discontinuity of $g(t)$ are $\frac{1}{4}$ and $-\frac{2}{3}$ that was already found previously.

Mark them on t axis:

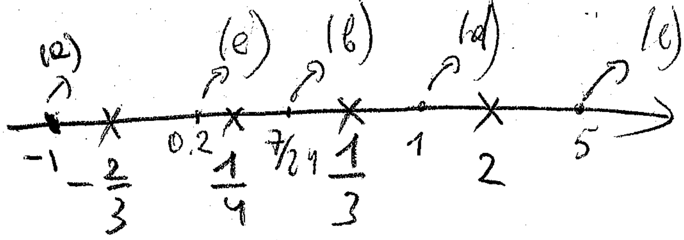


Fig. 1

Now for each item we need to find the maximal interval containing t_0 and not containing points in Fig. 1

(a) $t_0 = -1$: $t_0 < -\frac{2}{3} \Rightarrow$ the answer is $\boxed{t < -\frac{2}{3}}$

(b) $t_0 = 7/24$: $\frac{1}{4} < t_0 < \frac{1}{3} \Rightarrow$ the answer is $\boxed{\frac{1}{4} < t < \frac{1}{3}}$

(c) $t_0 = 5$: $2 < t_0 \Rightarrow$ the answer is $\boxed{t > 2}$

(d) $t_0 = 1$: $\frac{1}{3} < t_0 < 2 \Rightarrow$ the answer is $\boxed{\frac{1}{3} < t < 2}$

(e) $t_0 = 0.2 = \frac{1}{5}$: $-\frac{2}{3} < t_0 < \frac{1}{4} \Rightarrow$ the answer is $\boxed{-\frac{2}{3} < t < \frac{1}{4}}$

Problem 2 (a) (i) $y' = y^{1/5}$, $y(t_0) = y_0$

Let $f(t,y) = y^{1/5}$

$f(t,y)$ is continuous for every $(t,y) \in \mathbb{R}^2$ and

$\frac{\partial f}{\partial y}(t,y) = \frac{1}{5} y^{-4/5}$ is continuous everywhere except $y = 0$

So for every points (t_0, y_0) s.t. $\boxed{y_0 \neq 0}$

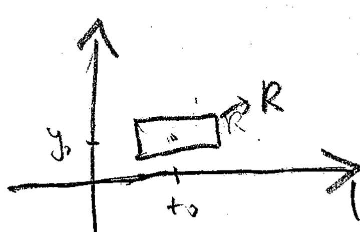


Fig 2

then exists a small rectangle R, not meeting the line $y = 0$ and containing (t_0, y_0) such that the assumptions of Theorem 2.4.2 of the book hold in R (see Fig 2) and therefore for such (t_0, y_0) there exists a unique solution of the corresponding initial value problem on an interval (t_0-h, t_0+h) for some $h>0$.

a (ii) $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$, $y(t_0)=y_0$

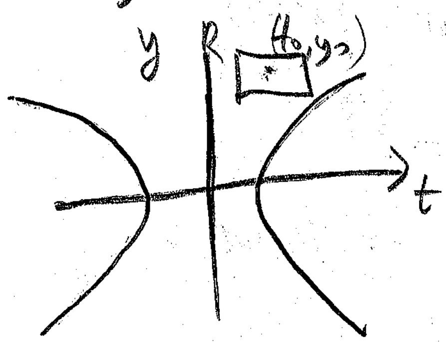
Let $f(t,y) = \dfrac{\ln|ty|}{1-t^2+y^2}$

$f(t,y)$ is continuous except (t,y) s.t.

$$ty = 0 \quad \text{or} \quad 1-t^2+y^2=0$$

or equivalently $\quad t=0$ or $y=0$ or $t^2-y^2=1$

(see Fig 3)



Note that if $ty \neq 0$ and $t^2-y^2 \neq 1$

then f is differentiable (because $\ln|ty|$ and $\dfrac{1}{1-t^2+y^2}$ are differentiable)

So for all (t_0, y_0) s.t. $\boxed{t_0 y_0 \neq 0 \text{ and } t_0^2 - y_0^2 \neq 1}$

all assumptions of the existence and uniqueness

Theorem 2.4.2

Theorem hold in some (maybe small) rectangle R, containing (t_0, y_0) and not meeting the sets $ty=0$ and $t^2-y^2=1$ (see Fig 3)

Therefore for such (t_0, y_0) there exists a unique solution of the corresponding initial value problem on an interval (t_0-h, t_0+h) for some $h>0$

Problem 2 (b)

By item (a) (i) y_0 must be equal to 0 (because for $y_0 \neq 0$ we have the uniqueness of IVP)

Take $t_0=0$ for simplicity.

Since $y_0=0$ is an equilibrium point

$y(t) \equiv 0$ is a solution of IVP.

On the other hand, we can try to find a solution in the form $y(t) = ct^{\alpha}$ then

$$y' = c\alpha t^{\alpha-1} \text{ and } y^{1/5} = c^{1/5} t^{\alpha/5} \Rightarrow$$

$$\alpha - 1 = \frac{\alpha}{5} \text{ and } c\alpha = c^{1/5} = \cdots$$

$$\frac{4}{5}\alpha = 1 \Rightarrow \alpha = \frac{5}{4} \quad \Rightarrow c^{4/5} = \frac{1}{\alpha} \Rightarrow c = \alpha^{-5/4} \left(\frac{4}{5}\right)^{5/4} \Rightarrow$$

$y(t) = \left(\frac{4}{5}t\right)^{5/4}$ is also a solution of the same IVP different from $y(t) \equiv 0$.

Remark One can also get the same solution $y(t)=\left(\frac{4}{5}t\right)^{5/4}$ by formally applying the method used for seperable equation (although to justify this method for the equation under consideration one has to assume that $y\neq 0$)

$$y' = y^{1/5} \xRightarrow{\text{seperate}} y^{-1/5}\,dy = dt \xRightarrow{\text{integrate}}$$

$$\frac{5}{4}y^{4/5} = t + C \Rightarrow y = \left(\frac{4}{5}t + \frac{4}{5}C\right)^{5/4}$$

For $C=0$ we get $y(t)=\left(\frac{4}{5}t\right)^{5/4}$ as in the previous arguments

Problem 3

$$y' = y^2 - 7y + 6 = (y-6)(y-1)$$

(a) Equilibrium points satisfy $(y-6)(y-1)=0 \Rightarrow$

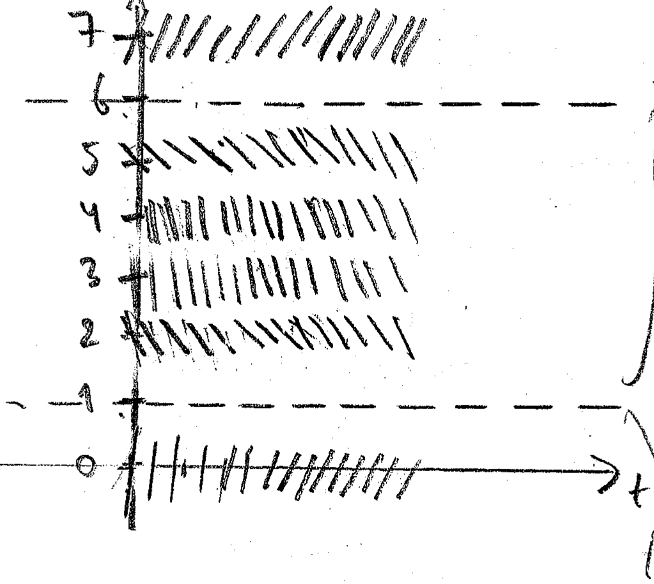
$y=1$ & $y=6$

slope	$+$	1	$-$	6	$+$

(b)

y	0	1	2	3	4	5	6	7
slope	6	0	−4	−6	−6	−4	0	6

see sketch on the next page



(c) i. $y(0)=4 \Rightarrow$ since $1<4<6$ and $y_1(t)\equiv 1$, $y_2(0)\equiv 6$ are solutions; then from the uniqueness the graphs of different solutions do not intersect $\Rightarrow$ $1<y(t)<6$ for any t $\Rightarrow$ since $f(y)=(y-6)(y-1)<0$ for $y\in(1,6)$, then $y'(t)<0$ for any t $\Rightarrow$ $y(t)$ is decreasing $\Rightarrow$ $\boxed{\lim_{t\to+\infty} y(t)=1}$ (the closest equilibrium point to $y_0=4$ as y decreases)

and $\boxed{\lim_{t\to-\infty} y(t)=6}$ (the closest equilibrium point to $y_0=4$ in the direction of increasing of y)

ii. Analyze all y_0 according to their location w.r.t. the equilibrium points 1 & 6:

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- If $1 < y_0 < 6$, then exactly by the same argument as in the previous subitem

$$\lim_{t\to+\infty} y(t) = 1$$

- If $y_0 = 1$, then $y(t) \equiv 1 \Rightarrow \lim_{t\to+\infty} y(t) = 1$

as for $y_0 = 4$.

- If $y_0 < 1$, then $y(t) < 1 \Rightarrow$ since $f(y) > 0$ for all $y < 0$, then $y'(t) > 0$ for all $t \Rightarrow y(t)$ is increasing $\Rightarrow \lim_{t\to+\infty} y(t) = 1$ as for $y_0 = 4$.
- If $y_0 = 6$, then $y(t) \equiv 6 \Rightarrow \lim_{t\to+\infty} y(t) = 6 \neq 1$

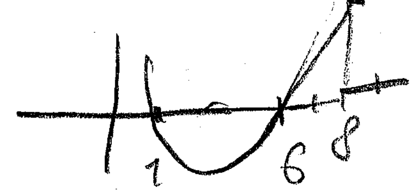
i.e <u>not</u> as for $y_0 = 6$

- If $y_0 > 6$, then $y(t) > 6 \Rightarrow \lim_{t\to+\infty} y(t)$ (if exists) is not equal to 1, so it is <u>not</u> as for $y_0 = 6$

Combining all case $\lim_{t\to+\infty} y(t) = 1 \Leftrightarrow \boxed{y_0 < 6}$

iii) If $y(0) = 8 > 6$, then $y(t) > 6$ for any t $\Rightarrow y'(t) > 0$ (because $f(y) > 0$ for $y > 6$) $\Rightarrow$ $y(t)$ is increasing.

Besides $f(y) \geq f(8) = 14$



for $y \geq 8 \Rightarrow$

$y'(t) \geq 14 \underset{\text{Integrate and use } y(0)=8}{\Longrightarrow} y(t) \geq 14t + 8 \xrightarrow[t\to+\infty]{} +\infty$ (see Fig. 4)

Therefore $\boxed{y(t) \to +\infty \text{ as } t \text{ increases}}$

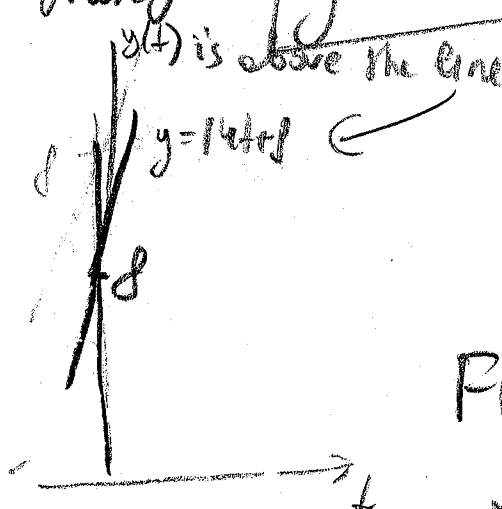


Fig. 4

(d) $y' = (y-6)(y-1) \to$ separable equation

$\frac{dy}{(y-1)(y-6)} = dt \Rightarrow \int \frac{dy}{(y-1)(y-6)} = t + C_1$

$\frac{1}{(y-1)(y-6)} = \frac{1}{5}\left(\frac{1}{y-6} - \frac{1}{y-1}\right) \Rightarrow$

$\frac{1}{5}\left(\ln|y-6| - \ln|y-1|\right) = t + C_1$

$\ln\left|\frac{y-6}{y-1}\right| = 5t + \underbrace{5C_1}_{C_2} \overset{\text{Exponentiate}}{\Longrightarrow} \frac{y-6}{y-1} = \underbrace{e^{C_2}}_{=C} e^{5t} \Rightarrow$

$$\frac{y-6}{y-1} = Ce^{5t} \Rightarrow y-6 = Ce^{5t}(y-1)$$

$$y(0)=8 \Rightarrow \quad \frac{8-6}{8-1} = C \Rightarrow C = \frac{2}{7} \Rightarrow$$

$$\frac{y-6}{y-1} = \frac{2}{7}e^{5t} \Rightarrow$$

Express y: $\quad y-6 = \frac{2}{7}e^{5t}(y-1) \Rightarrow$

$$\left(1-\frac{2}{7}e^{5t}\right)y = 6-\frac{2}{7}e^{5t} \Rightarrow y = \frac{6-\frac{2}{7}e^{5t}}{1-\frac{2}{7}e^{5t}} = \boxed{\frac{42-2e^{5t}}{7-2e^{5t}}}$$

The point of discontinuity is when $7-2e^{5t}=0 \Rightarrow t = \frac{1}{5}\ln\frac{7}{2}$

$\Rightarrow$ the solution of IVP is defined

for $\boxed{t < \frac{1}{5}\ln\frac{7}{2}}$

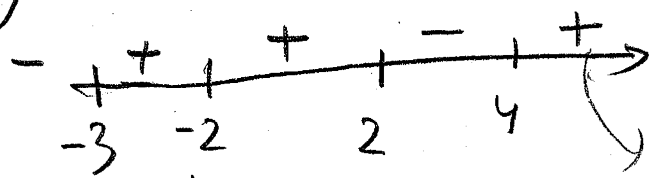
Problem 4 $\quad y' = \underbrace{(y^2-4)(y+2)(y^2-y-12)}_{f(y)}$

(a) $y^2-4 = (y-2)(y+2) \quad \Rightarrow \quad f(y) = (y-2)(y+2)^2(y-4)(y+3) = 0$

$y^2-y-12 = (y-4)(y+3)$

$\Rightarrow$ the equilibrium points are $\boxed{y = -3, -2, 2, 4}$

(b)

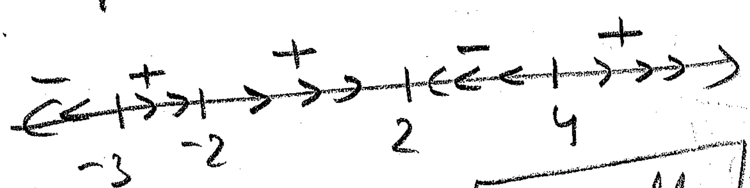


↔ signs of slopes

Because the coefficient of the higest degree term is >0

no change of the sign because in the factorization of $(y+2)^2$ appears

The phase line is :



(c) $y_0 = 4$ is $\boxed{\text{unstable}}$ (the sign of the slope changes from − to + as one passes y_0 in the positive direction

$y_0 = 2$ is $\boxed{\text{asymptotically stable}}$ (the sign of the slope changes from + to −)

$y_0 = -2$ is $\boxed{\text{semistable}}$ (the sign of the slope does not change)

$y_0 = -3$ is $\boxed{\text{unstable}}$ (the sign of the slope changes from − to +)

(11)

Problem 5 Rem (Problem 6 of Honors Section Bonus question) Actually this particular problem does not represent the approximative nature of the Euler method, because occasionally the Euler broken line in this particular case coinsides with the actual solution (I apologize it was not my intention)

(a)

	t	y	slope
step 0	0	0	1
step 1	0.2	0.2	1
step 2	0.4	0.4	1
step 3	0.6	0.6	1
step 4	0.8	0.8	1
step 5	1	1	1

$\Rightarrow \quad y_{\text{Euler with } h=0.2}(1) = \boxed{1}$

Find the actual solution: $y' + y = 1+t \to$ linear first order

Integrating factor $\mu' = \mu \Rightarrow \mu = e^t \Rightarrow (e^t y)' = (1+t)e^t \Rightarrow$

$e^t y = \int (1+t)e^t\,dt + C = \int e^t + \int te^t\,dt + C =$

$= \cancel{e^t} + te^t - \underbrace{\cancel{\int e^t\,dt}}_{e^t} + C = te^t + C \Rightarrow$

12) $y = t + Ce^{-t}$ $y(0)=0 \to C=0 \Rightarrow y(t)=t \Rightarrow$

$y(1)=1 \to$ coincides with the approximated value

(b) The answer exactly the same (occasionally for this initial conditions);

The slope on each step will be 1 and the Euler broken line is the graph of the actual solution $y(t)=t$.